On Reversible Markov Chains and Maximization of Directed Information

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***Abstract*—In this paper, we consider a dynamical system, whose state is an input to a memoryless channel. The state of the dynamical system is affected by its past, an exogenous input, and causal feedback from the channel's output. We consider maximizing the directed information between the input signal and the channel output, over all exogenous input distributions and/or dynamical system policies. We demonstrate that under certain conditions, reversibility of a Markov chain implies directed information is maximized. With this, we develop achievability theorems for channels with (infinite) memory as well as optimality conditions for sequential estimation of Markov processes through dynamical systems with causal feedback. We provide examples, which includes the exponential server timing channel and the trapdoor channel.**

I. INTRODUCTION

Time reversibility plays an important role in disciplines concerning dynamical systems, e.g. in physics (conservation laws); statistical mechanics (in terms of equilibrium states); stochastic processes (e.g. queuing networks [1] and convergence rates of Markov chains [2, ch 20]); and biology (e.g. trans paths in ion channels [3]). However, its use in providing information-theoretic fundamental limits appears to be somewhat limited [1]. In queuing systems, the celebrated Burke's theorem [1], [6] uses time reversibility to show that, in a certain stochastic dynamical system - an M/M/1 queue in steady-state - *the state of the system (queue) at time t is independent of all outputs (departures) before time t*. This observation has been used in proving achievability theorems for queuing timing channels [7], [8],[9]. In feedback information theory, statistical independence of one random variable at time t from others up to and including time t plays an important role - for example, tightness conditions in the converse to the channel coding with feedback [2]. This begs the question: *what might be the role of time reversibility in characterizing information-theoretic fundamental limits of stochastic systems with dynamics?*. Or in other words, what are the conditions when time reversibility is sufficient in characterizing fundamental limits through via implications of statistical independence? Recent developments at the intersection of information theory and control have demonstrated how *directed information* [12], [13], [14], [15] characterizes fundamental limitations of stochastic systems with dynamics. Here, we consider a class of stochastic systems for which time reversibility implies saturation of fundamental limits that are characterized by directed information.

[1]although Mitter et al [4], [5] have related Markov chain reversibility to entropy flow and equilibrium states in thermodynamic systems

[2]which is part of the "posterior matching principle" for optimal communication with feedback over a DMC [10, Sec III],[11]

A. Problem Setup

Consider a dynamical system with exogenous inputs $\underline{Z} = (Z_i \in \mathcal{Z} : i \geq 0)$ where the state $X_i \in \mathcal{X}$ is an input to a discrete memoryless channel (DMC) with output $Y_i \in \mathcal{Y}$:

$$P_{Y_i|X^i,Z^i}(y_i|x^i, z^i) = P_{Y|X}(y_i|x_i), \quad \forall i \geq 0. \qquad (1)$$

The *policy* $\underline{\mu} = \left(\mu_i : \mathcal{X} \times \mathcal{Z}^i \times \mathcal{Y}^{i-1} \to \mathcal{Y}, \quad i \geq 1\right)$ of the dynamical system governs how the state is affected by past exogenous inputs, and channel outputs:

$$x_1 \triangleq z_0; \qquad x_i = \mu_i\left(z^{i-1}, y^{i-1}\right), \quad i \geq 1 \qquad (2)$$

The statistics of all processes are governed by $P_{\underline{Z}}$, $P_{Y|X}$, and $\underline{\mu}$ - so we denote the *system* as $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$. See Figure 1. Examples of such systems in this setup could include:

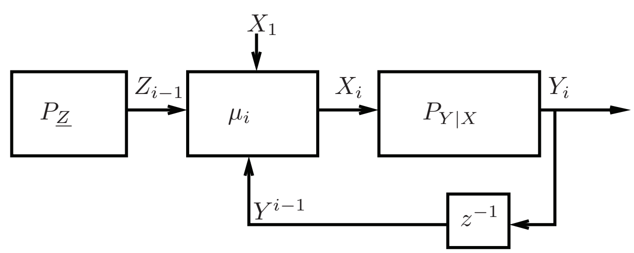

Fig. 1. A random process $\underline{Z} \sim P_{\underline{Z}}$ and dynamical system with policy $\underline{\mu}$.

- A queuing system, where X_i is the number of customers at time i in the queue, Z is the arrival process, Y is the departure process, and μ encodes the update laws of a queuing system's dynamics. Note that in the context of communication *without encoder feedback*, feedback appears because this is a channel with memory; it is the objective to $P_{\underline{Z}}$ (Section IV). Or, given a fixed Poisson process $P_{\underline{Z}}$ for the arrival process, the policy μ can be optimized, to minimize the difference distortion in timings between the input and output processes and minimize the average queue lengths (Section V).

- A biological system [16] where a 'transmitter' signals by adding molecules into a system. The receiver samples signals from the channel and withdraws a molecule, which in the process modifies the state (e.g. chemical concentration) of the system.
- A brain-machine interface [17], where Z is a Markov process representing a human's intended trajectory of some smooth path. X_i represents the subsequent 'intended' thought the user specifies to the interface. The interface observes a noisy version, Y_i. Note that here, the importance of the dynamical system is paramount: the interface visually (or otherwise) provides Y_i [3] back to the user - who uses a strategy (policy) $\underline{\mu}$ to guide the decoder's estimate in the same vicinity of the intended trajectory $\underline{Z}$. The objective is to develop an optimal strategy $\underline{\mu}$ and estimator to minimize distortion between the estimate $\hat{\underline{Z}}$ and the source $\underline{Z}$ (Section V).

Define the capacity-cost function [18] to be the maximum mutual information between the state process and the channel output subject to a 'stability' constraint on X:

$$\begin{aligned} C(\eta, P_{Y|X}, b) &\triangleq \max_{P_X \, s.t. \, \mathbb{E}[\eta(X)] \leq b} I(P_X, P_{Y|X}), \quad (3) \\ I(P_X, P_{Y|X}) &\triangleq I(X;Y) \end{aligned}$$

where $\eta : \mathcal{X} \to \mathbb{R}_+$ is the state cost function. The reason to specify a cost function η on the state X and not Z is two fold: (1) in control-like scenarios, (1a) Z need not be stationary and (1b) X is the state of some physical system that is desired to be stable; (2) in many communication scenarios, a cost function on X is equivalent to a cost function on Z (See Section VI). The directed information $I(Z^n \to Y^n)$ represents the 'causal' effect that the source process Z has on the channel output Y (which may differ from $I(Z^n; Y^n)$) and is given by [19]:

$$I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) \triangleq \frac{1}{n} I(Z^n \to Y^n) = \sum_{i=1}^{n} I(Z^{i-1}; Y_i | Y^{i-1}) \quad (4)$$

For the system $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$ in Fig 1, $I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$ is upper bounded by the capacity $C(\eta, P_{Y|X}, b)$ as follows:

$$\begin{aligned} I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) &= \frac{1}{n}\left[\sum_{i=1}^{n} H(Y_i|Y^{i-1}) - H(Y_i|Z^{i-1}, Y^{i-1})\right] \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} H(Y_i|Y^{i-1}) - H(Y_i|Z^{i-1}, Y^{i-1}, X_i)\right] \quad (5a) \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} H(Y_i|Y^{i-1}) - H(Y_i|X_i)\right] \quad (5b) \\ &\leq \frac{1}{n}\left[\sum_{i=1}^{n} H(Y_i) - H(Y_i|X_i)\right] \quad (5c) \\ &\leq C(\eta, P_{Y|X}, b) \quad (5d) \end{aligned}$$

[3]or some signal - say $\hat{Z}_i$ representative of Y_i

where (5a) follows from (2); (5b) follows from (1); (5c) follows because conditioning reduces entropy; and (5d) follows from (3). The equalities in (5c) and (5d) hold if and only if:

$$I(Y_i; Y^{i-1}) = 0; \ \Leftrightarrow I(X_i; Y^{i-1}) = 0 \quad (6)$$

$$P_{X_i} \sim \bar{P}_X(\eta, P_{Y|X}, b) \triangleq \arg\max_{P'_X} I(P'_X, P_{Y|X}) \quad (7)$$

Conditions in (6) are equivalent because of memorlyless property of the DMC (1). Thus the directed information of the system $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$ with state constraint is maximized if the conditions (6) and (7) are satisfied. (6) implies statistical independence of X_i with Y^{i-1} and is analogous to the statement made by Burke's theorem in Sec I - which was proven using reversible Markov chains. This indicates a possible connection between reversible Markov chains and saturating the fundamental limits of certain stochastic systems.

Section III provides the main theorem - that weds reversibility of Markov chains with maximization of directed information - by providing a sufficient condition for (5d) to be tight in terms of an underlying reversible Markov chain. We then begin to explore what the tightness of (5d) means within the context of the system $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$ described in Figure 1, by (i) providing the cost-constrained capacity for a class of infinite-memory communication channels with input Z and output Y, (Section IV); and (ii) providing a sufficient condition for optimality of linear-time-invariant estimator pairs for sequential estimation of Markov processes through dynamical systems with causal feedback (Section V). In both cases, time-reversibility is used to prove optimality. Section VI provides examples, which include the exponential server timing channel (ESTC) [7], [8] and the trapdoor channel [20],[21].

II. Definitions and Notations

- We write $Z \perp\!\!\!\perp W$ if Z and W are independent.
- Define $\mathcal{S}$ as the set of stationary Markov policies, i.e.

$$x_i = \mu_i\left(z^{i-1}, y^{i-1}\right) = \mu\left(x_{i-1}, z_{i-1}, y_{i-1}\right), \quad i \geq 1 \quad (8)$$

- A random process $\underline{W} = (W_i : i \geq 1)$ is a Markov chain with matrix P if $P_{W_{k+1}|W^k}\left(w_{k+1}|w^k\right) = P_{W_{k+1}|W_k}\left(w_{k+1}|w_k\right)$. If the Markov chain is positive Harris recurrent (PHR), then a unique invariant distribution π exists, which satisfies $\pi^T = \pi^T P$.
- If Z_1^∞ are i.i.d., then from (1) and (8), $\underline{X}$ is a Markov chain. Suppose $\underline{\mu} \in \mathcal{S}$. Define $\Psi\left(\mu, P_{Y|X}\right)$ to be the set of $P_{\underline{Z}}$ s.t. (a) Z_1^∞ are i.i.d. $\sim P_Z$ and independent of Z_0, (b) the subsequent Markov chain $\underline{X}$ is PHR, and (c) $Z_0 = X_1 \sim \pi \triangleq \pi(P_Z, \mu, P_{Y|X})$.
- Denote $\mathcal{R}(\mathcal{W})$ as the set of *reversible Markov chains* distributions on $\mathcal{W}$, namely the set of $P_{\underline{W}}$ for which (a) $\underline{W}$ is a PHR Markov chain, and (b) the forward and reverse time processes are statistically indistinguishable:

$$\begin{aligned} (W_j : 1 \leq j \leq n) &\overset{d}{=} (W_{n-j+1} : 1 \leq j \leq n) \quad (9a) \\ \Leftrightarrow \pi_i P_{ij} &= \pi_j P_{ij} \quad \forall i, j \in \mathcal{W} \quad (9b) \end{aligned}$$

III. Main Theorem

In this section, we will show that a policy with a linear, time-invariant Markov structure and i.i.d. exogenous source $P_{\underline{Z}}$ results in (5c)-(5d) being tight - provided an underlying Markov chain is reversible.

Definition 3.1: Denote $\mathcal{L}$ to be the set of systems $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$ s.t.:

(a) For some field $\mathbb{F}$, $\mathcal{Z} = \mathcal{X} = \mathcal{Y} = \mathbb{F}$ and μ satisfies:

$$\begin{aligned} x_i &= x_{i-1} + a_2 z_{i-1} + a_3 y_{i-1} && \text{(10a)} \\ &= \tilde{x}_{i-1} + a_2 z_{i-1} && \text{(10b)} \\ \tilde{x}_k &\triangleq x_k + a_3 y_k. && \text{(10c)} \end{aligned}$$

(b) $P_{\underline{Z}} \in \Psi\left(\mu, P_{Y|X}\right)$,

(c) $P_{\underline{X},\underline{\tilde{X}}} \in \mathcal{R}\left(\mathcal{X} \times \mathcal{X}\right)$,

(d) $\pi(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) \sim \bar{P}_X(\eta, P_{Y|X}, b)$

Note that $\tilde{x}$ in condition (a) is the update to the state *after* the output of the channel is taken into consideration and *before* the source z is updated to the state. Condition (b) means that the source distribution consists of i.i.d. inputs. Condition (c) is the key time reversibility condition on $(\underline{X}, \underline{\tilde{X}})$, which in many scenarios is equivalent to a reversibility condition on $\underline{X}$ (e.g. when $\underline{X}$ is a birth-death chain [22]). Lastly, condition (d) simply states that each $X_i \sim \bar{P}_X(\eta, P_{Y|X}, b)$. With this definition, we have the following theorem:

Theorem 3.2: If $(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) \in \mathcal{L}$, then $I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) = C(\eta, P_{Y|X}, b)$.

Proof: Following [6], [22], note from (10c) that:

$$\begin{aligned} y_k &= a_3^{-1}(\tilde{x}_k - x_k) && \text{(11)} \\ z_{k-1} &= a_2^{-1}(x_k - \tilde{x}_{k-1}) && \text{(12)} \end{aligned}$$

As such, we have

$$\begin{aligned} I\left(\tilde{X}_i; Y^{i-1}\right) &= I\left(\tilde{X}_i; (\tilde{X}_k - X_k):\ 1 \le k \le i\right) && \text{(13)} \\ &= I\left(\tilde{X}_i; (\tilde{X}_{k-1} - X_k):\ i+1 \le k \le 2i\right) && \text{(14)} \\ &= I\left(\tilde{X}_i; Z_i^{2i-1}\right) && \text{(15)} \\ &= 0, && \text{(16)} \end{aligned}$$

where (13) follows from (11); (14) from Defn. 3.1(d) and (9a); and (15) from (12); and (16) from Defn 3.1 (c). Thus $\tilde{X}_i \perp\!\!\!\perp Y^{i-1}$. From Defn 3.1 (c) and (10c), this implies that $X_i \perp\!\!\!\perp Y^{i-1}$. Also since $Z_i \sim \bar{P}_X(\eta, P_{Y|X}, b)$ from Defn 3.1(d), equations (6) and (7) are satisfied and thus $I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) = C(\eta, P_{Y|X}, b)$. ∎

IV. Fixing $\underline{\mu}$ and optimizing $P_{\underline{Z}}$

In this section, consider fixing $\underline{\mu} = \underline{\mu}^*$ of structure (10) and optimizing over $P_{\underline{Z}}$ to maximize $I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X})$, as in Figure 2. We demonstrate here that not only is the upper bound $I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) \le C(\eta, P_{Y|X}, b)$ tight when reversibility holds, but reliable communication from $\underline{Z}$ to $\underline{Y}$ - a channel with infinite memory - up to capacity $C(\eta, P_{Y|X}, b)$ is possible. The proof again uses the fact that reversibility implies statistical independence to invoke a SLLN for Markov chains.

Lemma 4.1: Consider communication over the infinite memory channel with $\underline{\mu}$ fixed, given by (10). If there exists a $P_{\underline{Z}}^* \in \Psi\left(\mu, P_{Y|X}\right)$ s.t. the system $(P_{\underline{Z}}^*, \underline{\mu}^*, P_{Y|X}) \in \mathcal{L}$, then the channel has capacity $C(\eta, P_{Y|X}, b)$.

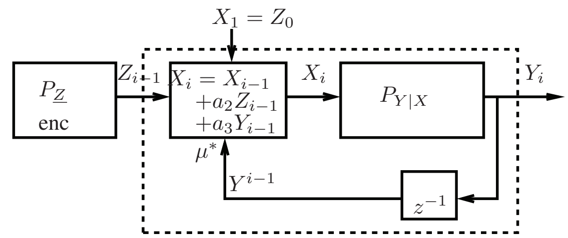

Fig. 2. Feedforward communication over a channel with memory (and internal feedback) with inputs Z and outputs Y.

Proof: To prove achievability, we use the information density. $i_{Z^n,Y^n}(z^n, y^n) \triangleq \log \frac{P_{Y^n|Z^n}(y^n|z^n)}{P_{Y^n}(y^n)}$ [23]:

$$\begin{aligned} & \frac{1}{n} \log i_{Z^n,Y^n}(Z^n, Y^n) \\ = & \frac{1}{n}\sum_{i=1}^n \log \frac{P_{Y_i|Y^{i-1},Z^{i-1},Z_i^n}\left(Y_i|Y^{i-1}, Z^{i-1}, Z_i^n\right)}{P_{Y_i|Y^{i-1}}\left(Y_i|Y^{i-1}\right)} \\ = & \frac{1}{n}\sum_{i=1}^n \log \frac{P_{Y_i|Y^{i-1},Z^{i-1}}\left(Y_i|Y^{i-1}, Z^{i-1}\right)}{P_{Y_i|Y^{i-1}}\left(Y_i|Y^{i-1}\right)} && \text{(17)} \\ = & \frac{1}{n}\sum_{i=1}^n \log \frac{P_{Y_i|Y^{i-1},Z^{i-1},X_i}\left(Y_i|Y^{i-1}, Z^{i-1}, X_i\right)}{P_{Y_i|Y^{i-1}}\left(Y_i|Y^{i-1}\right)} && \text{(18)} \\ = & \frac{1}{n}\sum_{i=1}^n \log \frac{P_{Y_i|X_i}\left(Y_i|X_i\right)}{P_{Y_i}\left(Y_i\right)} && \text{(19)} \\ \overset{a.s.}{\to} & I(\pi, P_{Y|X}) = C(\eta, P_{Y|X}, b) && \text{(20)} \end{aligned}$$

where (17) follows because the $Z_i^n \perp\!\!\!\perp Z^{i-1}$; (18) follows from from (2); (19) from (1) and the fact (Thm 3.2) that $Y_i \perp\!\!\!\perp Y^{i-1}$; and (20) follows from the Strong Law of Large Numbers for Markov Chains applied to $(W_i = (X_i, Y_i) : i \ge 1)$ [2].

The converse follows from a simple application of Fano's inequality as it pertains to the message $W \in \{1, \ldots, 2^{nR}\}$:

$$\begin{aligned} R &\le \epsilon_n + \frac{1}{n} I(W; Y^n) = \epsilon_n + I_n(P_{\underline{Z}}, \underline{\mu}, P_{Y|X}) \\ &\le \epsilon_n + C(\eta, P_{Y|X}, b). \end{aligned}$$

∎

V. Fixing P_Z and optimizing μ

In this section, we consider sequential estimation of a Markov processes through a dynamical systems with causal feedback. [4] Given an i.i.d. source $\tilde{Z}_0^\infty$ over a field $\mathbb{F}$ drawn $\sim P_Z$, define its Markov process representation as

$$Z_0 = \tilde{Z}_0, \qquad Z_i = Z_{i-1} + c_2 \tilde{Z}_i. \tag{21}$$

In this setting, we are interested in *tracking* $\underline{Z}$ as it is passed through a noisy channel. But the policy $\underline{\mu}$ gets to know the

[4] This appears to be related to the joint-source channel coding problem with feedback and bandwidth expansion given in [24, Sec. 4.2], but there are two differences: (a), the source Z here will be modeled as a (non-stationary) Markov process; (b) the distortion function will operate on the state, the reproduction, *and* the previous reproduction (to bring forth the nature of the dynamical system). These two considerations were explicitly developed to relate this to problems of decentralized of control [25], [12].

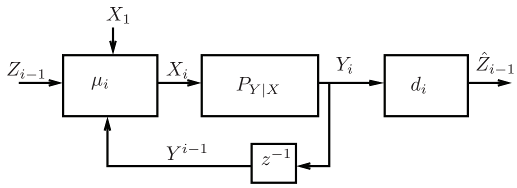

Fig. 3. sequential estimation of a Markov process through a dynamical system with causal feedback

output of the noisy channel, to help guide the *estimator* $\underline{d} = \left(d_i : \mathcal{Y}^i \to Z,\ i \geq 1\right)$. So we have two costs that we have to attempt to make low: (1) the cost of bad reproductions, represented with a time-invariant distortion measure given by $\rho : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ and (2): the state cost function that relates this to control and stability, measured by $\eta(x)$. So in short, our expected cost is given by

$$\rho_n(z^n, \hat{z}^n) = \frac{1}{n}\sum_{i=1}^{n} \rho(z_i, \hat{z}_i, \hat{z}_{i-1}) + \alpha\eta(x_i). \tag{22}$$

where $\alpha > 0$ balances the importance of control of X and estimation of Z. We must optimize the (*possibly non-linear, time-varying*) policy, decoder pair:

$$(\underline{d}^*, \underline{\mu}^*) = \arg\min_{\underline{d},\underline{\mu}} \mathbb{E}_{\underline{d},\underline{\mu}}[\rho_n(Z_1^n, \hat{Z}_1^n)] \tag{23}$$

$$\hat{Z}_n \triangleq d_n(Y^n). \tag{24}$$

The sequential rate distortion function is [12, Ch. 5]:

$$\overrightarrow{R}_n(D) \triangleq \inf_{P \in \overrightarrow{\mathcal{P}}_n(D)} \frac{1}{n} I(Z^n \to \hat{Z}^n) \tag{25}$$

$$\overrightarrow{\mathcal{P}}_n(D) \triangleq \Big\{ P_{\hat{Z}^n|Z^n} : \mathbb{E}[\rho_n(Z_1^n, \hat{Z}_1^n)] \leq D, \tag{26}$$

$$P_{\hat{Z}_i|\hat{Z}^{i-1},Z^n} = P_{\hat{Z}_i|\hat{Z}^{i-1},Z^{i-1}}\ \forall i \in \{1,\ldots,n\} \Big\} \tag{27}$$

(27) encodes the physical constraints of causal encoding/decoding imposed on each d_i. It is already known (c.f. [12, Lemma 5.4.2]) the minimizer of $\overrightarrow{R}_n(D)$ will have an added benefit: it will satisfy $P_{\hat{Z}_i|\hat{Z}^{i-1},Z^i} = P_{\hat{Z}_i|\hat{Z}^{i-1},Z_i}$ for any Markov process $\underline{Z}$. This hints at a possible time-invariant structure for the policy, estimator pair.

For any $\underline{\mu}$ and $\underline{d}$ s.t. $P_{\underline{X}}$ satisfies the stability constraint $\mathbb{E}[\eta(X)] \leq b$ and $P_{\hat{Z}^n|Z^n} \in \overrightarrow{\mathcal{P}}_n(D)$, then:

$$\begin{aligned} \overrightarrow{R}_n(D) &\leq \frac{1}{n} I(Z^n \to \hat{Z}^n) \\ &\leq \frac{1}{n} I(Z^n \to Y^n) && (28a) \\ &\leq C(\eta, P_{Y|X}, b) && (28b) \end{aligned}$$

(28a) follows from the data processing inequality, and (28b) follows from (5d) and the definition of $C(\eta, P_{Y|X}, b)$. Note that this holds with equality if and only if: (a) d_i is information-lossless; (b) $X_i \perp\!\!\!\perp Y^{i-1}$ for all i, and (c) $X_i \sim \bar{P}_X(\eta, P_{Y|X}, b)$.

A policy-estimator pair $(\underline{\mu}, \underline{d})$ is *LTI* if for some $\tilde{c}_3 \neq 0$:

$$x_i = z_i - \hat{z}_{i-1}, \hat{z}_0 = 0; \quad \hat{z}_i = \hat{z}_{i-1} + \tilde{c}_3 y_i, \ \ i \geq 1 \tag{29a}$$

From (29) and (21),

$$\begin{aligned} x_i &= x_{i-1} + (z_i - z_{i-1}) - (\hat{z}_{i-1} - \hat{z}_{i-2}) && (30a) \\ &= x_{i-1} + c_2\tilde{z}_i - c_3 y_{i-1} && (30b) \end{aligned}$$

and so (29a) is equivalent to (10) operating on the i.i.d. process $\tilde{Z}$ - so this machinery is equivalent to the stationary Markov policy setup in Section III.

Definition 5.1: We say an LTI policy-estimator pair $(\underline{\mu}^*, \underline{d}^*)$ are *optimal* if they minimize (22) for some $\alpha > 0$. In other words, this definition means no other policy-estimator pair $(\underline{\mu}, \underline{d})$ could result in smaller distortion without increasing expected state cost.

Theorem 5.2: Consider a Markov source $\underline{Z}$ defined as (21), DMC $P_{Y|X}$, fixed state cost function η and sequence of distortion functions $\underline{\rho}$. Then under the LTI policy-estimator pair $(\underline{\mu}, \underline{d})$ is optimal if for some $\alpha_1, \alpha_2 > 0$ and $\rho' : \mathcal{X} \to \mathbb{R}$:

(a.) the $\underline{Y}$ are i.i.d.
(b.) $\eta(x) = \alpha_1 D\left(P_{Y|X=x} \| P_Y\right) + \eta_0$
(c.) $\rho(z, \hat{z}, \hat{z}') = -\alpha_2 \log P_{Y|X}\left((\frac{\hat{z}-\hat{z}'}{\tilde{c}_3})|(z - \hat{z}')\right) + \rho'(z)$

Proof: Note that from (29), $\hat{Z}^i$ and Y^i form a bijection. By virtue of (29) and the DMC (1), it follows that

$$P_{\hat{Z}_i|\hat{Z}^{i-1},Z^n}\left(\hat{z}_i|\hat{z}^{i-1}, z^n\right) = P_{\hat{Z}_i|\hat{Z}^{i-1},Z^{i-1}}\left(\hat{z}_i|\hat{z}^{i-1}, z^{i-1}\right) \tag{31}$$

$$= P_{Y|X}\left((\frac{\hat{z}-\hat{z}'}{\tilde{c}_3})|(z - \hat{z}')\right) \tag{32}$$

where (31) follows from (27), which holds for any Markov source with causal encoder [12]; (32) follows from (29),(1). Note as a consequence that

$$\frac{1}{n}\sum_{i=1}^{n} \rho(z_i, \hat{z}_i, \hat{z}_{i-1}) = -\alpha_2 \log P_{Z^n,\hat{Z}^n}\left(z^n, \hat{z}^n\right) + \rho'(z^n) \tag{33}$$

$$\frac{1}{n}\sum_{i=1}^{n} \eta(x_i) = \alpha_0 D\left(P_{Y^n|X^n=x^n} \| P_{Y^n}\right) + \eta_0 \tag{34}$$

where the latter holds because Y^n are i.i.d. Thus, since Y^n are i.i.d., for this setting, $\overrightarrow{R}_n(D) = C(\eta, P_{Y|X}, b)$. The rest of the proof follows directly from [18]. ■

This leads to a corollary linking reversibility and optimality:

Corollary 5.3: If an LTI policy-estimator pair leads to $\underline{Z} \in \Psi\left(\mu, P_{Y|X}\right)$, then it is optimal with respect to η and ρ specified by Thm 5.2 conditions (b)-(c).

So *in these algebraic settings, the distortion measures that are sufficient for optimality are time-invariant.*

VI. Examples and Discussion

A. Gaussian Channels

Consider the problem in Section V with $\tilde{Z}_i$ are i.i.d. Gaussian, $\eta(x) = x^2$, an additive Gaussian channel, and squared error distortion. We recover the optimality of using an LTI state-estimator pair, with $\tilde{c}_3$ playing the role of the sequential MMSE estimator [25],[12, Chap 6] by invoking Theorem 5.2 directly.

B. Queuing Timing Channels

Let $\mathbb{F} = \mathbb{Q}$, $\eta(x) = +\infty 1_{\{x \notin \mathbb{Z} \cup x<0\}}$. Consider the policy $\underline{\mu}$ given by (10) with $c_2 = 1, \tilde{c}_3 = 1$ and the DMC with outputs $y \in \{0,1\}$ specified by the Z channel:

$$P_{Y|X}(1|x) = \begin{cases} 0 & x = 0 \\ \mu\Delta & x > 0 \end{cases} \quad (35)$$

where $\Delta \ll 1$. This represents the ESTC of rate μ: $Y_i = 1$ when a departure occurs, and Z_i represents the number of arrivals in the ith length-Δ time slot, and service times are exponentially distributed (see [7, Fig 4]). With Z_1^∞ i.i.d. with $\mathbb{P}(Z_i) = e^{-1}\mu\Delta$ and Z_0 a geometric variable of parameter e^{-1}, the subsequent birth-death Markov chain is stationary, and so Theorem 3.2 applies. This recovers the Poisson process of rate $\lambda = e^{-1}\mu$ as the optimal input distribution to the ESTC, and moreover, its capacity [5]. From this perspective, although an ESTC is nonlinear with infinite memory, it is composed of a memoryless channel and an algebraic, LTI, dynamical system with internal feedback. This leads to an alternative way to understanding its capacity.

Now consider Section V with $c_2 = 1$ in (21) and $\tilde{Z}_i \in \{0,1\}$ with $\mathbb{P}(Z_i = 1) = \lambda\Delta$. Then Z represents the counting function representation of a Poisson process. Note that with the appropriate "queuing distortion measure" [26] specified as in Theorem 5.2, note that something interesting follows: of all possibly dynamical system policies and estimators, selecting the policy-estimator pair $(\underline{\mu}, \underline{d})$ to be the dynamics of a queuing system (29) (with $\tilde{c}_3 = 1$ implying $\underline{\hat{Z}}$ is the counting function representation of the departure process $\underline{Y}$), is optimal. This also implies that uncoded transmission of a Poisson process over an ESTC is optimal.

Other extensions to queuing timing channels fit within this framework as well: see for example the variety of queuing systems in [22] for which joint reversibility holds. Another such queuing timing channel is an example of Gallager: 'the queue with feedback' [1, p 204-205]. Here, with probability $1 - p_0$ departures from the queue instantaneously return to the input of the queue (independent of all other processes). The 'effective' Z channel changes $\mu\Delta$ to $p_0\mu\Delta$ and all other arguments hold. In these cases, our cost-capacity theorem provides the capacity of such channels.

C. The Chemical (Trapdoor) Channel

Let $\mathbb{F} = \mathbb{Q}$, $\eta(x) = +\infty 1_{\{x \notin \{0,1,2\}\}}$. Consider the policy $\underline{\mu}$ given by (10) with $c_2 = 1, \tilde{c}_3 = 1$ and the DMC with outputs $y \in \{0,1\}$ specified by the 'inverted' Z channel:

$$P_{Y|X}(1|x) = \begin{cases} 0 & x = 0 \\ \frac{1}{2} & x = 1 \\ 1 & x = 2 \end{cases} \quad (36)$$

Here, the 'chemical channel' [16], [20], [21] has a state $X \in \{0,1,2\}$ represents the number of red balls in a system that can contain two balls, each of which is either red or blue. At each time, one of the two trapdoors is opened randomly, with $Y_=1$ if a red ball departs and 0 otherwise. Each entering ball Z_i is either red (1) or blue (0). With Z_1^∞ i.i.d. with $P(Z_i = 1) = p$, the three-state Markov chain is birth-death with steady-state distribution $\pi = [p^2, 2p(1-p), (1-p)^2]$ and $I(\pi, P_{Y|X}) = h_b(\pi_0 + \frac{1}{2}\pi_1) - \pi_1$. Thm 3.2 and reversibility imply that $R < I(\pi, P_{Y|X})\big|_{p=\frac{1}{2}} = 0.5$ is achievable - coinciding with [20].

[5]Also, if we let the state cost function $\eta(x) = 1$ for all non-negative integers and ∞ otherwise, then any $b \in (0,1)$, places a constraint on the queue utilization - given by $\frac{\lambda}{\mu}$. This equates to a constraint on the rate of the input process Z and recovers the rate-constrained capacity $C(\lambda,\mu) = \lambda \log \frac{\mu}{\lambda}$

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