An Optimizer's Approach to Stochastic Control Problems With Nonclassical Information Structures

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***Abstract*—We present a general optimization-based framework for stochastic control problems with nonclassical information structures. We cast these problems equivalently as optimization problems on joint distributions. The resulting problems are necessarily *nonconvex*. Our approach to solving them is through *convex relaxation*. We solve the instance solved by Bansal and Başar ("Stochastic teams with nonclassical information revisited: When is an affine law optimal?", *IEEE Trans. Automatic Control*, 1987) with a particular application of this approach that uses the data processing inequality for constructing the convex relaxation. Using certain f-divergences, we obtain a new, larger set of inverse optimal cost functions for such problems. Insights are obtained on the relation between the structure of cost functions and of convex relaxations for inverse optimal control.**

***Index Terms*— Optimal control, stochastic systems, optimization, information theory, decentralized control, networked control systems.**

I. Motivation and Contribution

This paper concerns the following stochastic control problem with nonclassical information structure: minimize

$$J(\gamma_0, \gamma_1) = \mathbb{E}\left[\kappa(S, X, Y, \widehat{S})\right], \qquad (\mathbf{N})$$
$$X = \gamma_0(S), \quad Y = \phi(X, W), \quad \widehat{S} = \gamma_1(Y)$$

where S, W are random variables with fixed statistics, ϕ is a fixed measurable function, $X, \widehat{S}$ are constrained by the *information structure*, i.e., X is adapted to S alone and $\widehat{S}$ is adapted to Y alone, and the decision variables are measurable functions γ_0, γ_1 (Fig. 1). In general, these problems are challenging to solve. For example, if $\kappa(S, X, Y, \widehat{S}) = (S - X)^2 + (X - \widehat{S})^2$, $Y = X + W$ and S, W are independent Gaussian, then this becomes the long standing open problem of Witsenhausen [3] about which many fundamental issues remain to be clarified.

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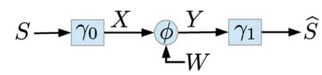

Fig. 1. Nonclassical information structure and the setting of $\mathbf{N}$.

In 1987, Bansal and Başar [2] solved a nontrivial problem with this information stucture. They consider the case where the problem is to minimize

$$\kappa(S, X, Y, \widehat{S}) = k_0 X^2 + s_{01} XS + (\widehat{S} - S)^2 \qquad (\mathbf{B})$$

where $Y = X + W$, $S \sim \mathcal{N}(0, \sigma_0^2), W \sim \mathcal{N}(0, \sigma_w^2)$ are independent Gaussian random variables and $k_0 > 0$ is a scalar constant. Their approach was to interpret Fig. 1 as a communication system where γ_0 and γ_1 are the encoder and decoder, respectively, and then use information-theoretic bounds to obtain a solution. An important aspect of their proof is that it relies critically on the information-theoretic data processing inequality (DPI). This remains the only known logic to proving optimality for problem $\mathbf{B}$ [4]. It is evident that ascertaining if there is some aspect of this proof that can be generalized to other instances of $\mathbf{N}$ would involve first understanding the role of the DPI in this problem.

Our objective in this paper is to show how certain optimization-based perspectives can be used to understand problems like $\mathbf{N}$. We show that these problems are essentially nonconvex and develop an approach for them based on *convex relaxation*. This approach yields a convex-analytic framework that subsumes previous information-theoretic results. The solution of $\mathbf{B}$, which was obtained in [2] by appealing to communication theory, is obtained as a special case of this convex relaxation approach by using the DPI for convexifying the problem. This viewpoint yields a broader interpretation of the role of the DPI in $\mathbf{B}$ and an explanation for what makes $\mathbf{B}$, which is prima facie only a slight modification of the Witsenhausen problem [2], tractable, even while the Witsenhausen problem itself remains unsolved.

A. Main New Results and Insights

This paper argues that common to all problems $\mathbf{N}$ is a fundamental nonconvexity that germinates from the information structure of these problems. We show that $\mathbf{N}$ is equivalent to an optimization problem where the expected cost is minimized over the joint distribution of all variables but where γ_0 and γ_1 are allowed to be *random*. This problem has a linear objective and necessarily *nonconvex* constraints. Analogous formulations for problems with classical information structure lead to *convex* optimization problems. Nonconvexity implies that a general purpose approach for solving $\mathbf{N}$ is not obvious. A possible solution strategy is to construct a convex relaxation with the same objective and a larger and convex feasible region. This problem being convex is potentially more accessible. The strategy then is to find a solution of the relaxation that is feasible for $\mathbf{N}$, which would thereby be a solution of $\mathbf{N}$.

We show that a particular convex relaxation can be constructed using the DPI and $\mathbf{B}$ can be solved using the above strategy with this relaxation. Implicit in the proof of Bansal and Başar is the use of the variational equations for the rate-distortion and capacity-cost function and their relation using the DPI. While we also use the DPI, in our proof *only its convexity properties are needed* in solving $\mathbf{B}$; no communication-theoretic properties are required. Thus the role of the DPI in $\mathbf{B}$ is only that of an artifice for convexifying the problem and as such the DPI may be replaced by any other suitable means of convexification. It is plausible that other convexifications would yield other solutions of $\mathbf{B}$ or solutions to other variants of $\mathbf{N}$.

Indeed such solutions are readily obtainable for the problem of *inverse optimal control* in which *cost functions*, for which a candidate policy is optimal, are sought. This problem is useful when one knows a policy adopted by controllers (suggested by, say, experimental data), and these controllers are known to be acting "optimally" according to some unknown criterion which one would like to determine (see, e.g., [5]). We show that the standard set of inverse optimal cost functions obtained via information-theoretic arguments correspond to those with the convexification using the DPI. However using other divergences, such as certain f-divergences, we obtain a new, larger set of inverse optimal cost functions. As such, these results provide a general purpose framework to propose other convexifications that will lead to similar formalisms.

Specifically for problem $\mathbf{B}$, our results amount to a new proof of optimality—our proof does not use the Cauchy–Schwartz inequality or parametrization (which are two of the steps in [2]). The tractability of $\mathbf{B}$ is explained by the fact that even though $\mathbf{B}$ is equivalent to a nonconvex optimization problem, it admits a known convex relaxation that is tight for it. Furthermore, we show that if a certain nondegeneracy condition holds, equality in the DPI is a *necessary condition* for optimality in $\mathbf{B}$. Consequently, any other proof of optimality of $\mathbf{B}$ must imply this equality, even though the DPI may not have been explicitly employed in the proof. This partially settles the open problem [4] of ascertaining whether there is a way of solving $\mathbf{B}$ that does not use the DPI.

Finally, by convex relaxation we give a lower bound on the Witsenhausen problem and a result that indicates that a certain "easy" solution approach cannot succeed for it.

B. Theoretical Explanation of the Results

There exists an optimization-based approach for Markov decision processes (stochastic control problems with *classical* information structures), where one optimizes the expected cost over distributions on state-action spaces, called *occupation measures* (see, e.g., [6], [7]). The relaxation of deterministic codes to random codes is a logical extension of this approach to problem $\mathbf{N}$. We minimize the expected cost over distributions that are consistent with the given marginals and that respect the information structure of the problem. But while the constraints in the classical information structure result in a convex (in fact, linear) program, in problem $\mathbf{N}$, the nonclassicality of the information structure implies the nonconvexity of the corresponding optimization problem.[1]

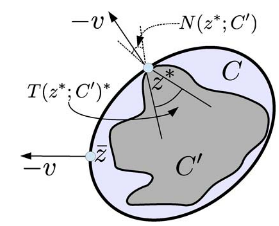

Fig. 2. Point z^* is optimal for both (2) and (1). $\bar{z}$ is optimal for (2) but is not feasible for (1).

Problem $\mathbf{B}$ is also nonconvex, but it happens to share its solution with a specific relaxation. To explain this, consider problems (1) and (2) below where v is a vector, C' is a nonconvex set and $C \supset C'$ is a convex set. The convex relaxation (2) of (1) is exact (or successful) if the solution of (2) lies within C'. This coincidence occurs in problem $\mathbf{B}$ if the set C is formed using the DPI. In general, an optimal solution of C may not lie in C' (e.g., point $\bar{z}$ in Fig. 2)

$$\min_{z\in C'} v^\top z \tag{1}$$

$$\min_{z\in C} v^\top z \tag{2}$$

The notion of *inverse optimality* seeks cost functions (the vector v) for which a candidate solution z^* is optimal; let and $\mathcal{I}(z^*;C')$ and $\mathcal{I}(z^*;C)$ be this set of functions for (1) and (2) for some $z^* \in C'$. z^* is a solution of (2) *if and only if* $-v$ belongs to $N(z^*;C)$, the *normal cone* of C at z^* (these are vectors "normal" to C and pointing outwards; see Fig. 2 and [8] for the details), whereby $\mathcal{I}(z^*;C) = -N(z^*;C)$. For v to be inverse optimal for (1), it is sufficient that $-v$ lies in $N(z^*;C')$ (the normal cone of C' at z^*, cf. Fig. 2), whereby $-N(z^*;C') \subseteq \mathcal{I}(z^*;C')$. Since $C \supset C'$ we have the relation

$$N(z^*;C) \subseteq N(z^*;C') \tag{3}$$

and thus, $\mathcal{I}(z^*;C) \subseteq \mathcal{I}(z^*;C')$. In summary, if $z^* \in C'$ solves (2), cost functions which are now inverse optimal for (2) are guaranteed to be inverse optimal for (1). In general this inclusion is strict. Nonconvexity of C' makes $N(z^*;C')$ hard to characterize. One can in general give only *necessary* conditions for inverse optimality for (1): e.g., KKT conditions yield a set of functions (denoted by $T(z^*;C')^*$, the *dual of the tangent cone*, in Fig. 2) that is larger than $\mathcal{I}(z^*;C')$. On the other hand, for the convex problem (2) the KKT conditions give $-N(z^*;C)$ exactly.

The relation (3) holds for any convex set C containing C'. Thus one gets a suite of inverse optimal cost functions by varying the convex set employed for creating the relaxation. In our case, the convexification with the DPI provides one set of inverse optimal cost functions for a variant of $\mathbf{B}$ (with $s_{01} = 0$). In fact it provides a stronger form of inverse optimality where "distortion-like" and "cost-like" terms are separately obtained.

[1]One may think of this as a different perspective on the observation that nonclassical problems like $\mathbf{N}$ are harder than those with classical information structure.

Suppose one asks for v_1 and v_2 where v_1, v_2 lie in given orthogonal subspaces such that $v = v_1 + v_2$ is inverse optimal for (1). v_1, v_2 are obtained as the projection of the normal cone along these subspaces, which can be obtained cleanly if C has a favorable structure. In $\mathbf{B}$ with $s_{01} = 0$, the DPI allows the inverse optimal distortion and cost to be separately unravelled. This is because of the agreement between the structure of the DPI and the structure of these terms.

By replacing the K-L divergence with other suitable f-divergences, one can vary the choice of the set C and obtain a larger set of inverse optimal cost functions, wherein f is also a parameter defining these functions.

The above convex analytic arguments are distinct from the information-theoretic arguments that have been applied for lower bounding problems like $\mathbf{N}$.[2] This latter argument involves embedding $\mathbf{N}$ into a problem that permits infinite delay (i.e., arbitrary block lengths) which is then subjected to information-theoretic analysis. This is somewhat unsatisfactory because, firstly, the appearance of mutual information is a likely consequence of the infinite block length problem, whereby the method is somewhat inflexible, and, second, the bounds will often be too loose (see Witsenhausen's discussion on this topic [10, Sec. 8] and Ho *et al.* [9]). Convex relaxation provides a broader paradigm for obtaining bounds for $\mathbf{N}$, and for problem $\mathbf{B}$, which is the most general problem we know of to have been solved by the above information-theoretic approach, the convex relaxation based bounds include the information-theoretic bounds as a special case (amounting to specific choice of the convexification).

C. Background and Organization

There are two main contributions from our work. First, convex relaxation can be seen more generally as an *approach* for problems with nonclassical information structure. Second, in the particular case of problem $\mathbf{B}$ our work clarifies certain issues, including its hardness and the role of the DPI in its solution. Given the limited space available here, we present some background relevant only to these aspects of our work.

Problem $\mathbf{N}$ is perhaps the simplest example of a stochastic control problem with a *dynamic* information structure [11] that is not *partially nested* [11]–[13]. The famous counterexample of Witsenhausen [3], a special case of $\mathbf{N}$, showed that LQG problems with the information structure of $\mathbf{N}$ need not have a linear optimal controller. This has prompted a line of research asking, "When is an optimal controller linear?" [14], and the result of Bansal and Başar [2] was one of the most general results along this direction. Prior to [2] it was known that for the Gaussian test channel (i.e., $\mathbf{B}$ with $s_{01} = 0$), the optimal codes are linear; in [2] they extended this to the case where $s_{01} \neq 0$. In later extensions they consider a version of $\mathbf{B}$ where the channel is vector-valued [15] and a system with dynamics and feedback [16]. In both problems [15], [16], the optimal controllers are found to be linear and in each of these proofs the DPI was used as a key ingredient. It remains an open problem to ascertain if there is another way of proving optimality for $\mathbf{B}$ that does not use the DPI [4]. In this context, our work shows that in the solution of $\mathbf{B}$ the DPI is only an artifice for convexifying the problem, and as such may be replaced by any other suitable convex set. However equality in the DPI is a necessary condition for optimality in $\mathbf{B}$, so any other proof must imply this equality.

Finding the optimal controller *within the class* of linear controllers for a decentralized LQG problem is also not straightforward; this problem can be cast as a minimization of the H_2-norm of the closed-loop map over all gain operators (matrices) that satisfy a *sparsity constraint* (the sparsity pattern is dictated by the information structure [17], [18]), and is often nonconvex. Over the past decade, there has been an explosion of results surrounding the concept of *quadratic invariance* [19]—an algebraic condition imposed on the sparsity pattern which allows the reformulation of this norm minimization as a convex problem. Fascinatingly, quadratic invariance was shown to be closely related to partially nested information structures [20], [21].

In problems like $\mathbf{N}$, a fundamental challenge is of understanding what makes these problems intractable. For example, one may conclude, that since the argument of γ_1 depends on the value of γ_0, and γ_1 does not have access to the argument of γ_0, problem $\mathbf{N}$ is essentially an optimization of $J(\gamma_0, \gamma_1(\gamma_0))$ over γ_0, γ_1 [11]. This is clearly a nonstandard problem in optimization over functions. Furthermore, it is a *nonconvex* problem in γ_0 whenever γ_1 is not linear or a suitable function [3], [11]. To quote Ho [11, p. 648], *"The computational as well as theoretical difficulties sketched above makes (P3) (i.e.,* $\mathbf{N}$*) a most difficult problem. It is still unsolved today, some 12 years*[3] *after Witsenhausen first analyzed it. In fact, ... we do not even have a sufficient condition for optimality for (P3) similar to the usual second variation condition in function optimization."*[4]

The above explanation for the hardness of $\mathbf{N}$ is somewhat unsatisfactory since it does not explain the differences in the degree of hardness of instances of $\mathbf{N}$. In particular, note that while we know little about the solution of the Witsenhausen problem, for the Bansal–Başar problem $\mathbf{B}$, which is a slight variant of the Witsenhausen problem, we do know the solution. This raises the natural question: what makes the Bansal–Başar problem tractable? To the best of our knowledge, an answer to this question from the viewpoint of optimization over functions is not known. Bansal and Başar [2] have made the observation that the presence of a term in "$\gamma_0\gamma_1$" makes the Witsenhausen problem hard. However this does not yet explain what makes $\mathbf{B}$ tractable.

Our work shows that $\mathbf{N}$ is equivalent to a nonconvex optimization problem in the space of joint distributions. Importantly, this result does not rely on the structure of the cost κ, but is a consequence of the information structure of $\mathbf{N}$. The tractability of $\mathbf{B}$ is then explained by the fact that, although it is a nonconvex problem, there happens to be a convex relaxation that is tight for it.

A closely related paradigm where one optimizes over joint distributions that are constrained to be consistent with given

[2] To quote Ho *et al.* [9, p. 307, footnote 5] *"There exist no general sufficiency conditions to verify the optimality of a solution (of a problem like* $\mathbf{N}$*) besides the Shannon bounds."*

[3] Now, 46 years....

[4] On another note, we recall the effort of Papadimitriou and Tsitsiklis [22] to understand the hardness of the Witsenhausen problem through the lens of computational complexity.

marginals is the subject of optimal transport theory [23]. The framework presented here may be seen as optimal transport with *additional constraints* which arise from the information structure. Recent work of Wu and Verdú [24] also casts the problem as an optimization over distributions, but the eventual problem they address has a different formulation from ours. Their focus is specifically on the Witsenhausen problem; they exploit the quadratic structure of the cost function and their newly discovered properties of the MMSE estimator [25]. With this, they can relate the Witsenhausen problem to a standard optimal transportation problem with quadratic cost, to characterize properties of an optimal solution.

At the heart of problems like $\mathbf{N}$ is the interplay of information and control [26], signalling and the dual role of control [4]. These problems have inspired an ongoing search for an understanding of the role of information in control [27], [28]. Indeed, a related line of research attempts to do dynamic programming for problems with various information structures [21], [29].

This paper is organized as follows. In Section II, we recall some preliminaries, the proof of Bansal and Başar and the communication-theoretic approach to problems like $\mathbf{N}$. Section III contains the optimization formulation, convex relaxation and the solution of $\mathbf{B}$. Section IV concerns inverse optimal control and Section V concerns the convex relaxation of the Witsenhausen problem. We conclude in Section VI.

II. Preliminaries

In this paper, by probability "distributions" of random variables we mean (joint) densities if the random variables are continuous and probability mass functions if they are discrete. We use lower case letters to denote specific values of random variables, which are denoted by upper case latters. For a distribution Q,"$Q_\bullet$" denotes the marginal of "$\bullet$." Whenever required, we abbreviate "$Q_V(v)$" as "$Q(v)$." Let P, Q be any two probability distributions on a finite set $\mathcal{X}$. The K-L divergence of P and Q is defined as $D(P\|Q) := \sum_{x\in\mathcal{X}} P(x)\log(P(x)/Q(x))$; in this paper $\log$ will denote the natural logarithm. If the underlying random variables are continuous and P, Q are densities, then $D(P\|Q) := \int_{\mathcal{X}} \log(P(x)/Q(x))P(x)dx$. For random variables X, Y with joint distribution Q the mutual information of X and Y is defined as $I(X;Y) := D(Q\|Q_X Q_Y)$.

If W, X, Y form a Markov chain, we write it succinctly as $W \to X \to Y$. Mutual information satisfies the data processing inequality for Markov chains [30]

$$I(W;X) \geq I(W;Y). \tag{4}$$

Consequently, if $W \to X \to Y \to Z$, then $I(X;Y) \geq I(W;Z)$. Finally, for an optimization problem $\mathbf{P}$ we denote the system of KKT conditions for this problem by $\mathrm{KKT}_{\mathbf{P}}$ and the optimal value by $\mathrm{OPT}(\mathbf{P})$.

A. The Problem of Bansal and Başar

We begin by recalling the proof from [2]. Bansal and Başar interpret $\mathbf{B}$ as a problem of minimizing the distortion $(\widehat{S} - S)^2$ in a communication system subject to soft constraints on power X^2 and XS. Their proof [2, Sec. III] uses the following series of arguments:

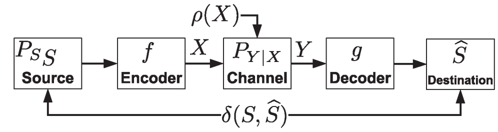

Fig. 3. Communication system with performance metrics.

1) The DPI (4) implies $I(S;\widehat{S}) \leq I(X;X+W)$.
2) Standard results about Gaussian channels give that, under the constraint $\mathbb{E}[X^2] \leq P^2$

$$\frac{1}{2}\log\frac{\sigma_0^2}{\mathbb{E}\left[(\widehat{S}-S)^2\right]} \leq I(S;\widehat{S}) \leq I(X;X+W)$$
$$\leq \frac{1}{2}\log\frac{P^2+\sigma_w^2}{\sigma_w^2}. \tag{5}$$

Combining these gives a lower bound on the distortion: $\mathbb{E}[(\widehat{S}-S)^2] \geq \sigma_0^2\sigma_w^2/(P^2+\sigma_w^2)$.

3) Cauchy–Schwartz inequality gives a lower bound on the third term $s_{01}\mathbb{E}[XS] \geq -|s_{01}|P\sigma_0$.

Thus the optimal value of $\mathbf{B}$, J^* is bounded below as $J^* \geq \sigma_0^2\sigma_w^2/(P^{*2}+\sigma_w^2) + k_0 P^{*2} - |s_{01}|P^*\sigma_0$, where P^* satisfies

$$(2k_0P^* - |s_{01}|\sigma_0)\left(P^{*2}+\sigma_w^2\right)^2 = 2P^*\sigma_0^2\sigma_w^2. \tag{6}$$

Finally, this bound is shown to be tight by the explicit construction of maps γ_0^*, γ_1^* for which $J(\gamma_0^*(S), \gamma_1^*(X+W))$ equals the lower bound. The most important step here is obtaining the lower bound on the distortion. The only known approach for this is through the DPI.

B. Communication-Theoretic Explanation

We now explain the proof of Bansal and Başar by comparing problem $\mathbf{B}$ with a problem from communication theory. We use the notation of Fig. 3. A source generates symbols (random variables) S taking values in a space $\mathcal{S}$, with probability $P_S(S)$. An encoder f maps these symbols to a space $\mathcal{X}$. Each symbol $X \in \mathcal{X}$ passes through a channel, which produces a symbol $Y \in \mathcal{Y}$ with probability $P_{Y|X}(Y|X)$. A decoder g maps Y to a destination space $\widehat{\mathcal{S}}$. The system is endowed with two performance metrics: a *distortion measure* $\delta : \mathcal{S}\times\widehat{\mathcal{S}} \to [0,\infty)$ and a *cost* $\rho : \mathcal{X} \to [0,\infty)$.

Standard communication theory employs *block codes* wherein multiple symbols are coded together. An n-block encoder (resp., decoder) is a map $f : \mathcal{S}^n \to \mathcal{X}^n$ (resp., $g : \mathcal{Y}^n \to \widehat{\mathcal{S}}^n$). A *single-letter code* is a 1-block code. The subject of rate-distortion theory is the minimization of average distortion and average cost, over the class of codes of arbitrary block length. Only in certain exceptional cases does it turn out that the optimal code is a single-letter code. Instead of seeking such cases, one may ask the inverse question studied by Gastpar *et al.* [31]: when is a given single-letter code Pareto optimal for average distortion and cost, over all codes of all block lengths?

For every single-letter code one can find a class of such *inverse optimal* cost and distortion functions such that the code is optimal for them in the above sense. To characterize this class, Gastpar uses the communication-theoretic constructs of the *rate-distortion* function R and *capacity-cost* function C. For each $D \geq 0$, $R(D)$ is given by

$$R(D) = \min_{p(\widehat{s},s):\mathbb{E}\left[\delta(S,\widehat{S})\right] \leq D,\ p(s)=P_S(s)} I(S,\widehat{S}). \tag{7}$$

For each $P \geq 0$, the capacity-cost function is

$$C(P) = \max_{p(x,y):\mathbb{E}[\rho(X)] \leq P,\ p(y|x)=P_{Y|X}(y|x)} I(X,Y). \tag{8}$$

If $D = (1/n)\sum_k \mathbb{E}[\delta(S_k,\widehat{S}_k)]$ and $P = (1/n)\sum_k \mathbb{E}[\rho(X_k)]$ are the average distortion and cost incurred by an n-block code, then $R(D) \leq C(P)$. This follows from the DPI (4). Gastpar argues that, except in some degenerate cases, $R(D^*) = C(P^*)$, is *equivalent* to the Pareto optimality of average distortion and cost $D = D^*$, $P = P^*$. (cf. Lemma 1, [31]).

If $R(D^*) = C(P^*)$, any *Pareto optimal single-letter code* f, g induces distributions $f(S) \sim p^*(x)$ and $g(Y)|S \sim p^*(\widehat{s}|s)$ where $p^*(x)$ and $p^*(\widehat{s}|s)$ achieve the optima in (8), (7), respectively. The corresponding inverse (Pareto) optimal cost and distortion are given by [31, Lemma 3, 4]

$$\begin{aligned}\rho(x) &= c_1 D\left(P_{Y|X}(\cdot|x)\|p_Y^*(\cdot)\right) + \rho_0 + \beta(x)\mathbb{I}_{\{p^*(x)>0\}}\\ \delta(s,\widehat{s}) &= -c_2 \log p^*(s|\widehat{s}) + d_0(s)\end{aligned} \tag{9}$$

where $c_1, c_2 > 0$, $\rho_0 \in \mathbb{R}$, $d_0 : \mathcal{S} \to \mathbb{R}$ is arbitrary, $\beta : \mathcal{X} \to [0,\infty)$ and p_Y^* is the marginal of Y under the distribution $p^*(x)P_{Y|X}(y|x)$. We recall that this characterization is also known from Csiszar and Körner [32].

1) Relation to the Bansal-Başar Problem: The key distinction between the information-theoretic problem described above and the control problem $\mathbf{N}$ is that the latter assumes a fixed (effectively, unit) block length. Since the controllers γ_0, γ_1 in problem $\mathbf{N}$ act on a single sample of their respective arguments, they are single-letter codes. One may consider an "information-theoretic" version of $\mathbf{N}$ in which the code allows infinite delay (i.e., arbitrary block lengths) and the objective is the average cost. The achievability results from rate-distortion theory when applied to this problem do not in general provide for the existence of a single-letter code (recall the discussion in [10] and [9]).

The cost in $\mathbf{B}$ is a sum of a distortion-like term $(u_1 - x_0)^2$, a cost-like term $k_0 u_0^2$ and a generalized cost term $s_{01} u_0 x_0$. Suppose $s_{01} = 0$. In this case problem $\mathbf{B}$ asks for a single-letter code that minimizes a sum of distortion and cost *over the class of all single-letter codes*. Clearly, if a single-letter code exists that is Pareto optimal in the sense of Gastpar, it is (upon appropriate scaling of cost functions) also optimal for $\mathbf{B}$. Problem $\mathbf{B}$ with $s_{01} = 0$ happens to be the exceptional case where such a code does exist. Indeed in (9) if f is taken to be the identity and $\widehat{S} = g(Y) = (\sigma_0^2/(\sigma_0^2 + \sigma_w^2))Y$, we get $\rho(x) = c_1 x^2 + \rho_0$ and $\delta(s,\widehat{s}) = c_2(s-\widehat{s})^2 + d_0(s)$ [31], which matches the distortion and cost terms in $\mathbf{B}$.

With this understanding the appearance of the data processing inequality in the proof of Bansal and Başar can be explained as a vestige of the communication-theoretic problem that allows infinite delay and a stronger notion of optimality that seeks optimality of a single letter code over codes of arbitrary block lengths.

Importantly, if, to obtain inverse optimal cost functions for the control problem $\mathbf{N}$, one argues via Gastpar's notion of inverse optimality, then the DPI is indispensable: a single-letter code is optimal in the sense of Gastpar *if and only if* it satisfies the characterizations in (9). The optimization approach we develop in the following sections is comparatively more flexible, and includes in it as a special case, the bound/inverse optimality characterizations obtained above.

III. The Optimization-Based Approach

This section contains the main contributions of this paper. In Section III-A, we formulate the problem $\mathbf{N}$ from Section I as a (nonconvex) optimization problem over joint distributions. In Section III-B we introduce its convex relaxation. In Section III-C, we solve problem $\mathbf{B}$ via its convex relaxation.

A. Problem Formulation

For the rest of the paper, we will consider the notation of Fig. 3. Presently, we assume for simplicity that $\mathcal{S}, \mathcal{X}, \mathcal{Y}, \widehat{\mathcal{S}}$ are finite. There are four random variables in the problem: $S, X, Y, \widehat{S}$; let Q be their joint distribution. To be a distribution of $S, X, Y, \widehat{S}$, Q must satisfy the following "constraints."

1) Q must be a distribution.
2) Q must be consistent with the *given marginals*. i.e.,

$$Q_S(s) \equiv P_S(s), \quad \text{and} \quad Q_{Y|X}(y|x) \equiv P_{Y|X}(y|x).$$

3) Q must respect the *information structure*, i.e., Q must satisfy Markovianity:

$$S \to X \to Y \to \widehat{S}.$$

Therefore, a "feasible" distribution Q is one that can be expressed as

$$Q(s,x,y,\widehat{s}) \equiv P_S(s)Q(x|s)P_{Y|X}(y|x)Q(\widehat{s}|y). \tag{10}$$

We use $\mathcal{Q}$ to denote the set of all such Q.

$$\begin{aligned}\mathcal{Q} := \Big\{Q | Q \in \mathcal{P}(\mathcal{Z}),\ &\text{for which } \exists\, Q_{X|S} \in \mathcal{P}(\mathcal{X}|\mathcal{S}),\\ &Q_{\widehat{S}|Y} \in \mathcal{P}(\widehat{\mathcal{S}}|\mathcal{Y}),\ \text{such that } Q \text{ satisfies (10)}\Big\}\end{aligned} \tag{11}$$

where $\mathcal{P}(\cdot)$ is the set of probability distributions on "$\cdot$" and we denote $\mathcal{Z} := \mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \widehat{\mathcal{S}}$.

Any $Q \in \mathcal{Q}$ is parametrized by the kernels $Q_{X|S}$ and $Q_{\widehat{S}|Y}$ (this parametrization has been studied in [33]). In the context of Fig. 3, these kernels can be thought of as "random" (single-letter) encoder and decoder, respectively. They together constitute what we call a *random code*.

Definition 3.1: A random encoder (decoder) is a conditional distribution $Q_{X|S}$ (resp., $Q_{\widehat{S}|Y}$) on $\mathcal{X}$ given a symbol in $\mathcal{S}$ (resp., on $\widehat{\mathcal{S}}$ given a symbol in $\mathcal{Y}$). A pair of a random encoder and decoder is a *random code*. An encoder (resp., decoder) is *deterministic* if and only if there is a function f (resp., g) such that $Q_{X|S}(x|s) = \mathbb{I}_{\{f(s)=x\}}$ for all x, s (resp., $Q_{\widehat{S}|Y}(\widehat{s}|y) = \mathbb{I}_{\{g(y)=\widehat{s}\}}$ for all $y, \widehat{s}$). A pair of deterministic encoder and decoder is a *deterministic code*.

The problem that minimizes cost $\mathbb{E}[\kappa(S, X, Y, \widehat{S})]$ over random codes is our optimization-based formulation for $\mathbf{N}$:

$$\min_{Q \in \mathcal{Q}} \langle \kappa, Q \rangle, \tag{$\mathbf{N}$}$$

where $\langle \cdot, \cdot \rangle$ denotes the operation $\langle \kappa, Q \rangle := \sum_z \kappa(z) Q(z)$ and z is a short-hand for a tuple $(s, x, y, \widehat{s}) \in \mathcal{Z}$. There is no abuse in denoting this problem as $\mathbf{N}$ since it has the same optimal value as $\mathbf{N}$ from Section I.

Lemma 3.1: The relaxed stochastic control problem over random codes (i.e., $\mathbf{N}$) has a solution that is a deterministic code.

Proof: Write $\mathbf{N}$ as

$$\min \sum_{s,x,y,\widehat{s}} \kappa(s, x, y, \widehat{s}) P_S(s) Q_{X|S}(x|s) P_{Y|X}(y|x) Q_{\widehat{S}|Y}(\widehat{s}|y)$$
$$\text{s.t. } Q_{X|S} \in \mathcal{P}(\mathcal{X}|\mathcal{S}), Q_{\widehat{S}|Y} \in \mathcal{P}(\widehat{\mathcal{S}}|\mathcal{Y})$$

$\mathbf{N}$ is thus a separably constrained bilinear optimization problem. It is a well known property of such a problem (see Exercise 4.25, [34]) that it has a solution that is an extreme point; in this case a deterministic code. ∎

Without loss of generality we consider this $\mathbf{N}$ as our stochastic control problem.

Remark III.1. Bilinearity: Note that the bilinearity of the objective of $\mathbf{N}$ is a consequence of its information structure; it is not clear if this property will hold for joint distributions representing more exotic information structures. □

B. Convex Relaxation

We now formulate the convex relaxation of $\mathbf{N}$, first noting the nonconvexity of this problem.

Lemma 3.2: $\mathcal{Q}$ is not a convex set.

Proof: Suppose $\mathcal{Q}$ is convex and consider distributions $Q^1, Q^2 \in \mathcal{Q}$ such that $Q^1_{X|S} \neq Q^2_{X|S}$ and $Q^1_{\widehat{S}|Y} \neq Q^2_{\widehat{S}|Y}$. Such Q^1, Q^2 exist so long as $\mathcal{X}$ and $\widehat{\mathcal{S}}$ are not both singletons. Let $t \in (0, 1)$. Then $q := tQ^1 + (1-t)Q^2$ belongs to $\mathcal{Q}$ and satisfies (10). Consequently, we must have $q(\widehat{s}|y) \equiv q(\widehat{s}|s, x, y)$. Therefore using (10)

$$q(\widehat{s}|y) = Q^1(\widehat{s}|y) + \frac{(1-t)Q^2(x|s)\left[Q^2(\widehat{s}|y) - Q^1(\widehat{s}|y)\right]}{tQ^1(x|s) + (1-t)Q^2(x|s)}$$

for s, x, y such that $q(s, x, y) > 0$. The right-hand side is independent of x, s for all $\widehat{s}, y$ if and only if one of the following is true: a) $Q^1(\widehat{s}|y) \equiv Q^2(\widehat{s}|y)$ or b) $(1-t)Q^2(x|s)/(tQ^1(x|s) + (1-t)Q^2(x|s)) = c$, a constant, for all x, s. The first possibility is ruled out by the choice of Q^1 and Q^2. The second possibility implies $t(Q^1(x|s) - Q^2(x|s)) \equiv 0$, which is ruled out since $t > 0$. Thus $\mathcal{Q}$ is not convex. ∎

Remark III.2: In the classical information structure Q would admit the decomposition

$$Q(s, x, y, \widehat{s}) \equiv P_S(s) Q(x|s) P_{Y|X}(y|x) Q(\widehat{s}|s, x, y)$$

instead of (10). It is easy to see that the set of such Q is convex. □

Thus, $\mathbf{N}$ is a nonconvex optimization problem with linear objective and nonconvex constraints. A possible approach for finding a global solution is to solve a convex relaxation of $\mathbf{N}$ with the hope of finding a solution of the relaxation that is feasible for $\mathbf{N}$.

A natural choice for the relaxation would be the convex hull of $\mathcal{Q}$; unfortunately this set has no known representation. We consider the following relaxation: we minimize $\langle \kappa, Q \rangle$ over all Q that satisfy the DPI and are consistent with the given marginals. This problem is denoted $\mathbf{C_N}$:

$$\min_{Q \in \mathcal{Q}_{\mathrm{DPI}}} \langle \kappa, Q \rangle \tag{$\mathbf{C_N}$}$$

where $\mathcal{Q}_{\mathrm{DPI}}$ is the set

$$\mathcal{Q}_{\mathrm{DPI}} = \Big\{ Q \in \mathcal{P}(\mathcal{Z}) | Q_S = P_S, Q_{Y|X} = P_{Y|X} \text{ and } I(Q_{X,Y}) \geq I(Q_{S,\widehat{S}}) \Big\} \tag{12}$$

and $I(\cdot)$ is the mutual information under distribution "$\cdot$." Clearly, $\mathcal{Q} \subset \mathcal{Q}_{\mathrm{DPI}}$ (since Markovianity implies the DPI (4), but is not equivalent to it) and $\mathcal{Q}_{\mathrm{DPI}}$ is convex (this follows easily from the proof of Lemma 3.3 below).

To solve $\mathbf{C_N}$, we write it as a mathematical program using additional variables $\{a(\widehat{s}|s) : \widehat{s} \in \widehat{\mathcal{S}}, s \in \mathcal{S}\}$, $\{b(x) : x \in \mathcal{X}\}$.

$$\begin{array}{lllr}
\mathbf{C_N} & \displaystyle\min_{Q,a,b} \sum_{s,x,y,\widehat{s}} \kappa(s,x,y,\widehat{s}) Q(s,x,y,\widehat{s}) & & \\
& \displaystyle\sum_{x,y} Q(z) = a(\widehat{s}|s) P_S(s), & \forall s, \widehat{s}, & : \lambda^a(s,\widehat{s}), \\
& \displaystyle\sum_{s,y,\widehat{s}} Q(z) = b(x), & \forall x, & : \lambda^b(x), \\
& I(aP_S) \leq I(P_{Y|X} b), & & : \lambda, \\
& \displaystyle\sum_{s,\widehat{s}} Q(z) = P_{Y|X}(y|x) \sum_{s,y,\widehat{s}} Q(z) & \forall x, y, & : \lambda^p(x,y), \\
\text{s.t.} & \displaystyle\sum_{\widehat{s}} a(\widehat{s}|s) = 1, & \forall s, & : \mu^a(s) P_S(s), \\
& a(\widehat{s}|s) \geq 0, & \forall s, \widehat{s}, & : \nu^a(\widehat{s}|s) P_S(s), \\
& \displaystyle\sum_x b(x) = 1, & & : \mu^b, \\
& b(x) \geq 0, & \forall x, & : \nu^b(x), \\
& Q(z) \geq 0, & \forall z, & : \nu(z).
\end{array}$$

Here, $I(aP_S)$ is the mutual information of $S, \widehat{S}$ under distribution $a(\widehat{s}|s)P_S(s)$ (and likewise $I(P_{Y|X} b)$). λ^a, λ^b, λ, μ^a, μ^b, ν^a, ν^b, and ν are (scaled) Lagrange multipliers corresponding to the specific constraints. Assume the existence of an interior (or Slater) point [35]. Then (Q, a, b) solves $\mathbf{C_N}$ if and only if

(Q, a, b) are feasible for $\mathbf{C_N}$ and there exist Lagrange multipliers that satisfy the following system of KKT conditions

$$\lambda^a(s,\widehat{s}) = \lambda \log \frac{a(\widehat{s}|s)}{a(\widehat{s})} + \mu^a(s) - \nu^a(\widehat{s}|s), \qquad (\mathbf{KKT_{C_N}})$$

$$\lambda^b(x) = -\lambda\left(D\left(P_{Y|X}(\cdot|x)\|b_Y(\cdot)\right) - 1\right) + \mu^b - \nu^b(x),$$

$$\nu(z) = \kappa(z) + \lambda^a(s,\widehat{s}) + \lambda^b(x) + \lambda^p(x,y) - \sum_{\bar{y}} \lambda^p(x,\bar{y}) P_{Y|X}(\bar{y}|x), \qquad (*)$$

$$\lambda \geq 0,\ I(P_{Y|X}b) - I(aP_S) \geq 0,\ \lambda\left(I(P_{Y|X}b) - I(aP_S)\right) = 0$$

$$\langle a(\cdot|s), \nu^a(\cdot|s)\rangle = 0,\ \nu^a(\widehat{s}|s) \geq 0,\ a(\widehat{s}|s) \geq 0,$$

$$\langle b, \nu^b\rangle = 0,\ \nu^b(\cdot) \geq 0,\ b(\cdot) \geq 0,$$

$$Q(\cdot) \geq 0,\ \nu(\cdot),\ \langle \nu, Q\rangle = 0$$

$\forall z \in \mathcal{Z}$, where we have used,

$$\partial I(aP_S)/\partial a(\widehat{s}|s)) = P_S(s)\log(a(\widehat{s}|s)/a(\widehat{s})),$$

$$\partial I(P_{Y|X}b)/\partial b(x) = D(P_{Y|X}(\cdot|x)\|b_Y(\cdot)) - 1$$

and $b_Y(\cdot)$ denotes $\sum_x P_{Y|X}(y|x)b(x)$.

To solve $\mathbf{N}$ for optimality or inverse optimality by the logic of the convex relaxation, we have to tackle $(\mathbf{KKT_{C_N}})$. The hardness of this depends on structure of κ and the choice of the convexification, as we explain below.

Remark III.3. DPI and the Structure of κ: Perhaps the most difficult part of $(\mathbf{KKT_{C_N}})$ is $(*)$, which contains all variables involved and the left-hand side ν has to be both nonnegative and complementary to Q. One way of dealing with this would be to set $\nu(\cdot) \equiv 0$. Then $(*)$ necessitates that κ be of the form

$$k(s,x,y,\widehat{s}) \equiv f_1(x) + f_2(s,\widehat{s}) + f_3(x,y) \qquad (13)$$

where the functions f_1, f_2, f_3 are determined by the Q that solves $\mathbf{C_N}$. The communication setting, where $\kappa(z) \equiv \delta(s,\widehat{s}) + \rho(x)$, agrees naturally with this structure and therefore taking $\nu(\cdot) \equiv 0$ may solve $(\mathbf{KKK_{C_N}})$. Furthermore, *inverse* optimality is easy for this structure (in fact, $\nu(\cdot) \equiv 0$ gives the same form of inverse optimal δ and ρ as (9)). Problem $\mathbf{B}$, on the other hand, has $\kappa(z) \equiv \delta(s,\widehat{s}) + \rho(x) + \tau(x,s)$ and does not agree directly with (13). In our solution of $\mathbf{B}$, we will not take $\nu(\cdot) \equiv 0$. The Witsenhausen problem has $\kappa(z) \equiv \zeta(x,\widehat{s}) + \tau(x,s)$ and it is even harder to reconcile with (13) (see Theorem 5.2). The mismatch between κ and (13) makes inverse optimality for these problems hard too.

The structure of the right-hand side of (13) is due the DPI constraint. Other convexifications would yield other structures and may allow greater ease in solving $(\mathbf{KKT_{C_N}})$. What we have developed is thus a general framework of which the DPI-based convexification is a particular application. □

In the following sections, we show that this view of the DPI as a convexification tool is sufficient for solving $\mathbf{B}$. In Section III-C1 we solve a variant of $\mathbf{B}$ with $\kappa = \delta + \rho$. In Section III-C2 we solve $\mathbf{B}$.

C. Solution of Specific Instances

1) Solution With Gastpar-Type Cost Functions: In this section consider a special case of $\mathbf{N}$, denoted $\mathbf{G}$, where $\kappa(z) \equiv \delta(s,\widehat{s}) + \rho(x)$:

$$\min_{Q\in\mathcal{Q}} \langle \delta + \rho, Q\rangle, \quad (\mathbf{G}) \qquad \min_{Q\in\mathcal{Q}_{\mathrm{DPI}}} \langle \delta + \rho, Q\rangle. \qquad (\mathbf{C_G})$$

and $\mathbf{C_G}$ is the convex relaxation of $\mathbf{G}$. We first write $\mathbf{C_G}$ as a mathematical program in a simpler form than $\mathbf{C_N}$. Notice that the variable "Q" in $\mathbf{C_N}$ can be dropped since the objective of $\mathbf{C_G}$ can be expressed in terms of a, b alone, and for any (a,b) in the set $\{(a,b)|a \in \mathcal{P}(\widehat{\mathcal{S}}|\mathcal{S}), b \in \mathcal{P}(\mathcal{X}), I(aP_S) \leq I(P_{Y|X}b)\}$, there exists Q satisfying the constraints in $\mathbf{C_N}$ (take $Q(z) \equiv a(\widehat{s}|s)P_S(s)b(x)P_{Y|X}(y|x)$). An equivalent form of $\mathbf{C_G}$ is thus the following:

$\mathbf{C_G}$ $\min_{a,b}$	$\sum_{s,\widehat{s}} \delta(s,\widehat{s})a(\widehat{s}\|s)P_S(s) + \sum_x \rho(x)b(x)$		
	$\sum_{\widehat{s}} a(\widehat{s}\|s) = 1,$	$\forall s,$	$: \mu^a(s)P_S(s),$
	$\sum_x b(x) = 1,$		$: \mu^b,$
s.t.	$I(aP_S) \leq I(P_{Y\|X}b),$		$: \lambda$
	$a(\widehat{s}\|s) \geq 0,$	$\forall s, \widehat{s}$	$: \nu^a(\widehat{s}\|s)P_S(s),$
	$b(x) \geq 0,$	$\forall x,$	$: \nu^b(x)$

Lemma 3.3: $\mathbf{C_G}$ is a convex optimization problem.

Proof: The objective is linear. All constraints except the third are linear. It is well known [30] that for fixed P_S, $I(aP_S)$ is convex in a and for fixed $P_{Y|X}$, $I(P_{Y|X}b)$ is concave in b, whereby the feasible region of $\mathbf{C_G}$ is convex. ■

Let $\mu^a, \mu^b, \lambda, \nu^a, \nu^b$ be the Lagrange multipliers of the constraints of problem $\mathbf{C_G}$. a, b are optimal for $\mathbf{C_G}$ if and only if together with the Lagrange multipliers, they satisfy

$$\delta(s,\widehat{s}) = -\lambda \log \frac{a(\widehat{s}|s)}{a(\widehat{s})} - \mu^a(s) + \nu^a(\widehat{s}|s) \qquad (\mathbf{KKT_{C_G}})$$

$$\rho(x) = \lambda\left(D\left(P_{Y|X}(\cdot|x)\|b_Y(\cdot)\right) - 1\right) - \mu^b + \nu^b(x)$$

$$\lambda \geq 0,\ I(P_{Y|X}b) - I(aP_S) \geq 0,\ \lambda\left(I(P_{Y|X}b) - I(aP_S)\right) = 0$$

$$\sum_{\widehat{s}} a(\widehat{s}|s) = 1,\ \langle a(\cdot|s), \nu^a(\cdot|s)\rangle = 0,\ \nu^a(\widehat{s}|s) \geq 0,$$

$$\sum_x b(x) = 1,\ \langle b, \nu^b\rangle = 0,\ \nu^b(\cdot) \geq 0,\ b(\cdot) \geq 0,\ a(\widehat{s}|s) \geq 0,$$

$\forall z \in \mathcal{Z}$. By making this simplification, we have effectively taken $\nu(\cdot) \equiv 0$, $\lambda^p(\cdot) \equiv 0$ in $(\mathbf{KKT_{C_N}})$ and in $(*)$ we are separately setting terms in $(s,\widehat{s})$ and x to be zero.

Remark III.4. Alternatives to Lagrange Multipliers: We note here that variational equations for the rate-distortion function were derived without invoking Lagrange multipliers for abstract alphabets by Csiszár [36]. We can derive our conditions through these results when the cost is of the type in $\mathbf{C_G}$. In this case, we may as above consider the problem

$$\min_{P_a\in\mathcal{L}_a, P_b\in\mathcal{L}_b} g_1(P_a) + g_2(P_b) \quad \text{s.t. } I(P_a) \leq I(P_b), \qquad (14)$$

where P_a and P_b are joint distributions on $(s, \widehat{s})$ and (x, y), respectively, and $\mathcal{L}_a$ and $\mathcal{L}_b$ are sets enforcing P_S to be fixed and $P_{Y|X}$ to be fixed and P_a, P_b to be joint distributions; finally $g_1(P_a) = \mathbb{E}[\delta(S, \widehat{S})]$ and $g_2(P_b) = \mathbb{E}[\rho(X)]$. In this formulation, infinite constraints are left in abstract form and Lagrange multipliers are invoked for the scalar constraint.

Now assuming the minimum is attained, there exists a $\lambda \geq 0$ such that the following problem has the same optimal solution as (14):

$$\min_{P_a \in \mathcal{L}_a, P_b \in \mathcal{L}_b} g_1(P_a) + g_2(P_b) + \lambda \left(I(P_a) - I(P_b) \right). \quad (15)$$

Assume that $\lambda > 0$ and define $\eta = 1/\lambda$. Problem (15) is separable, and is equivalent to solving the problems

$$\min_{P_a \in \mathcal{L}_a} I(P_a) + \eta g_1(P_a) \quad (16)$$

$$\text{and} \quad \max_{P_b \in \mathcal{L}_b} I(P_b) - \eta g_2(P_b). \quad (17)$$

The variational equations for solving (16) are given by [36, Lemma 1.4 and equation 1.21]

$$\log(dP_{\widehat{S}|S=s}/dP_{\widehat{S}})(\widehat{s}) = \log \alpha(s) - \eta \delta(s, \widehat{s})$$

where $\alpha(\cdot) \geq 1$ satisfies $\int_s \alpha(s) e^{-\eta\rho(s,\widehat{s})} \times P_S(ds) = 1, \forall \widehat{s}$. Likewise for (17), the variational equations are given by [37], $D(P_{Y|X=x} \| P_Y) - s\rho(x) + \chi(x)\mathbb{I}_{\{P_X(x)=0\}} = V$, where $\chi(\cdot) \geq 0$ and V is a scalar. When the cost is not "separable" in this sense there are, to the best of our knowledge, no known results that can be leveraged. The structure of the cost is a recurring theme in this paper (cf. Remark III.3) and points to the need for a development of a theory similar to that of [36] for more general costs. Further discussion on these conditions is contained in Remark IV.8. □

We are now prepared to demonstrate our goal which was to find a solution of $\mathbf{C_G}$ that is feasible for $\mathbf{G}$, in order to arrive at a solution to $\mathbf{G}$. Although we have taken $\mathcal{Z}$ to be finite, we assume that $(\mathbf{KKT_{C_G}})$ holds for this problem too. The arguments used are essentially geometric and infinite constraints can approached as in Luenberger [35].

Below we abbreviate $\xi_0(\gamma_0) := \sigma_0^2 \sigma_w^2 / ({\gamma_0}^2 \sigma_0^2 + \sigma_w^2)^2$, $\xi_1(\gamma_0) := \gamma_0 \sigma_0^2 / ({\gamma_0}^2 \sigma_0^2 + \sigma_w^2)$ and $\bar{P}(\gamma_0) := (\gamma_0^2 \sigma_0^2 + \sigma_w^2)/\gamma_0^2 \sigma_0^2$.

Theorem 3.1: Consider problem $\mathbf{B}$ with $s_{01} = 0$, i.e., consider $\mathbf{G}$ where $S \sim \mathcal{N}(0, \sigma_0^2)$, $Y = X + W$, $W \sim \mathcal{N}(0, \sigma_w^2)$ and independent of S, $\delta(s, \widehat{s}) \equiv (\widehat{s} - s)^2$ and $\rho(x) \equiv k_0 x^2$. Assume that a solution of $\mathbf{C_G}$ is characterized by the KKT conditions $(\mathbf{KKT_{C_G}})$. Then the PDF given by $(S, \gamma_0^* S, Y, \gamma_1^* Y)$ where γ_0^* satisfies $k_0 = \xi_0(\gamma_0^*)$ and $\gamma_1^* = \xi_1(\gamma_0^*)$ solves the convex relaxation $\mathbf{C_G}$. Furthermore, it is a solution of $\mathbf{G}$.

Proof: We parametrize $X = \gamma_0 S$ and $\widehat{S} = \gamma_1 Y$ and find γ_0 and γ_1 to solve $(\mathbf{KKT_{C_G}})$. The distribution obtained would be feasible for $\mathbf{G}$ since $S \to \gamma_0 S \to Y \to \gamma_1 Y$, and would thus be a solution of $\mathbf{G}$. With these constants we get (details can be found in [38, p. 51])

$$D\left(P_{Y|X}(\cdot|x) \| b_Y(\cdot)\right) = c_1 + \frac{x^2}{2\gamma_0^2 \sigma_0^2 \bar{P}} \quad (18)$$

$$\log \frac{a(\widehat{s}|s)}{a(\widehat{s})} = \frac{-1}{2\gamma_1^2 \sigma_w^2 \bar{P}} [\widehat{s} - \bar{P}\gamma_0\gamma_1 s]^2 + c_2(s) \quad (19)$$

where $\bar{P} = \bar{P}(\gamma_0)$, c_1 independent of x and c_2 is independent of $\widehat{s}$. To solve system $(\mathbf{KKT_{C_G}})$, take $\nu^b(x), \nu^a(\widehat{s}|s) \equiv 0$, $\mu^b = \lambda c_1 - \lambda$ and $\mu^a(s) \equiv -\lambda c_2(s)$. The first two equations in $(\mathbf{KKT_{C_G}})$ reduce to the identities

$$k_0 x^2 \equiv \lambda \frac{x^2}{2\gamma_0^2 \sigma_0^2 \bar{P}}, \quad (\widehat{s} - s)^2 \equiv \frac{\lambda}{2\gamma_1^2 \sigma_w^2 \bar{P}} [\widehat{s} - \bar{P}\gamma_0\gamma_1 s]^2.$$

Comparing coefficients, we get $k_0 = \xi_0(\gamma_0), \gamma_0\gamma_1\bar{P} = 1$ and $\lambda = 2\gamma_1^2 \sigma_w^2 \bar{P}$. Simplifying, we get $\gamma_0 = \gamma_0^*$, where γ_0^* satisfies $k_0 = \xi_0(\gamma_0^*)$ and $\gamma_1 = \gamma_1^* = \xi_1(\gamma_0^*)$. To show that this solves $(\mathbf{KKT_{C_G}})$, we need to show complementarity slackness of the DPI constraint. The values γ_0^* and γ_1^* imply $\lambda > 0$. Therefore, we need to show that equality holds in the DPI. This follows by noting that $\mathbb{E}[X^2] = {\gamma_0^*}^2 \sigma_0^2$ and $\mathbb{E}[(\widehat{S} - S)^2] = \sigma_w^2 \sigma_0^2 / ({\gamma_0^*}^2 \sigma_0^2 + \sigma_w^2)$, which means that the inequalities in (5) are tight with P^2 taken as ${\gamma_0^*}^2 \sigma_0^2$. Thus, the distributions induced by taking $X = \gamma_0^* S$ and $\widehat{S} = \gamma_1^* Y$ solve $(\mathbf{KKT_{C_G}})$ and are optimal for $\mathbf{C_G}$. Since they satisfy Markovianity, they also solve $\mathbf{G}$. ■

Remark III.5. Gaussian Test Channel: That these calculations look familiar and that the gains γ_0^* and γ_1^* are the well-known codes for the Gaussian test channel is not surprising; when the objective is a function of a, b alone, $(\mathbf{KKT_{C_N}})$ reduces to $(\mathbf{KKT_{C_G}})$, the right-hand side of which has the same form as Gastpar's (9). But recall that our calculations do not pertain to a communication problem. They are for the solution of a convex relaxation of a nonconvex *optimization* problem which was convexified using the DPI. □

γ_0^* and γ_1^* agree with the solution of Bansal and Başar for $s_{01} = 0$, but the important consequence, which is part of the message of this paper, is that, problem $\mathbf{B}$ (with $s_{01} = 0$) shares its solution with a specific convex relaxation. We emphasize this through Theorem 3.2 below.

Theorem 3.2: Let the setting and the assumptions of Theorem 3.1 hold. The optimal value of problem $\mathbf{B}$ with $s_{01} = 0$ equals the optimal value of its convex relaxation with $\mathcal{Q}_{\mathrm{DPI}}$. Furthermore, any solution of $\mathbf{B}$ with $s_{01} = 0$ solves its convex relaxation with $\mathcal{Q}_{\mathrm{DPI}}$.

2) Solution of the Bansal-Başar Problem: We now return to problem $\mathbf{B}$. $\mathbf{B}$ has the following structure: $\kappa(z) \equiv \delta(s, \widehat{s}) + \rho(x) + \tau(x, s)$, where $\tau(x, s) \equiv k\sqrt{\rho(x)}\tau'(s)$. We provide two proofs for the solution of $\mathbf{B}$. The first proof uses a reasoning similar to that used by Bansal and Başar—specifically, the use of the Cauchy–Schwartz inequality for bounding the term $\tau(x, s)$. In our second proof (which is our new proof), we do not use this inequality and argue directly.

Let $\mathbf{C_B}$ be the convex relaxation of problem $\mathbf{B}$

$$\min_{Q \in \mathcal{Q}} \langle \delta + \rho + \tau, Q \rangle \quad (\mathbf{B}) \quad \min_{Q \in \mathcal{Q}_{\mathrm{DPI}}} \langle \delta + \rho + \tau, Q \rangle. \quad (\mathbf{C_B})$$

By Cauchy–Schwartz inequality, we get $k\sum_{x,s} \sqrt{\rho(x)}\tau'(s) P_S(s) Q(x|s) \geq -\alpha(\sum_x \rho(x) b(x))^{1/2}$, where $\alpha = |k|\sqrt{\sum_s (\tau'(s))^2 P_S(s)}$. The use of the Cauchy–Schwartz inequality allows for the construction of an auxiliary problem, denoted $\mathbf{B}'$, where $\langle \tau, Q\rangle$ is replaced with $-\alpha\sqrt{\langle \rho, Q\rangle}$

$$\min_{Q\in\mathcal{Q}} \langle \delta+\rho, Q\rangle - \alpha\sqrt{\langle \rho, Q\rangle}. \qquad (\mathbf{B}')$$

The convex relaxation of $\mathbf{B}'$ using $\mathcal{Q}_{\rm DPI}$ is the following,

$$\min_{Q\in\mathcal{Q}_{\rm DPI}} \langle \delta+\rho, Q\rangle - \alpha\sqrt{\langle \rho, Q\rangle}. \qquad (\mathbf{C}_{\mathbf{B}'})$$

We will use $\mathbf{C}_{\mathbf{B}'}$ to solve $\mathbf{B}$. Note that $\mathbf{C}_{\mathbf{B}'}$ is *not* a special case of $\mathbf{C}_{\mathbf{N}}$ because its objective is not linear in Q, but it can be written as a mathematical program, as $\mathbf{C}_{\mathbf{G}}$ was, in terms of the variables a, b. The resulting formulation of $\mathbf{C}_{\mathbf{B}'}$ has the same constraints as $\mathbf{C}_{\mathbf{G}}$ and its objective is

$$\sum_{s,\widehat{s}} \delta(s,\widehat{s})P_S(s)a(\widehat{s}|s) + \sum_x \rho(x)b(x) - \alpha\left(\sum_x \rho(x)b(x)\right)^{\frac{1}{2}}.$$

Lemma 3.4: $\mathbf{C}_{\mathbf{B}'}$ is a convex optimization problem.

Proof: It suffices to show $\psi(b) := -\sqrt{\sum_x \rho(x)b(x)}$ is a convex function of b (cf. Lemma 3.3). The Hessian of ψ, $\nabla^2\psi(b) = (1/4)(1/(\sum_x \rho(x)b(x))^{3/2})\vec{\rho}\vec{\rho}^{\top}$, where $\vec{\rho} := (\rho(x))_{x\in\mathcal{X}}$ is the vector of values of ρ. Clearly, $\nabla^2\psi(b) \succeq 0$ for all b. ■

Consequently, the KKT conditions of $\mathbf{C}_{\mathbf{B}'}$ characterize its solution. We will use this to solve $\mathbf{B}$.

Theorem 3.3: Consider the problem of Bansal and Başar, i.e., consider problem $\mathbf{B}$ where $S \sim \mathcal{N}(0,\sigma_0^2)$, $Y = X + W$ where $W \sim \mathcal{N}(0,\sigma_w^2)$ and independent of S, $\delta(s,\widehat{s}) \equiv (\widehat{s}-s)^2$, $\rho(x) \equiv k_0x^2$, and $\tau(x,s) \equiv s_{01}xs$. Suppose the solution of $\mathbf{C}_{\mathbf{B}'}$ is characterized by its KKT conditions. Then the PDF $(S, \gamma_0^*S, Y, \gamma_1^*Y)$ where $\gamma_1^* = \xi_1(\gamma_0^*)$ and γ_0^* is a positive solution of

$$(2k_0\gamma_0^*\sigma_0 - |s_{01}|\sigma_0)\left(\gamma_0^{*2}\sigma_0^2 + \sigma_w^2\right)^2 = 2\gamma_0^*\sigma_0^3\sigma_w^2 \qquad (20)$$

solves $\mathbf{B}'$ and its convex relaxation $\mathbf{C}_{\mathbf{B}'}$. Furthermore, the PDF $(S, \gamma_0^{**}S, Y, \gamma_1^{**}Y)$ where $\gamma_1^{**} = \xi_1(\gamma_0^{**})$ and $\gamma_0^{**} = -\mathrm{sgn}(s_{01})\gamma_0^*$ solves $\mathbf{B}$.

Proof: By Cauchy–Schwartz inequality and since $\mathcal{Q} \subset \mathcal{Q}_{\rm DPI}$ we have

$$\mathrm{OPT}(\mathbf{B}) \geq \mathrm{OPT}(\mathbf{B}') \geq \mathrm{OPT}(\mathbf{C}_{\mathbf{B}'}).$$

We first show that the second inequality is tight by finding a solution of $\mathbf{C}_{\mathbf{B}'}$ that is feasible for $\mathbf{B}'$. Then we will construct a feasible point of $\mathbf{B}$ for which the objective of $\mathbf{B}$ equals $\mathrm{OPT}(\mathbf{B}')$.

The KKT conditions of $\mathbf{C}_{\mathbf{B}'}$ are similar to those of $\mathbf{C}_{\mathbf{G}}$, except for the equation

$$\rho(x)\left(1-\alpha/(2R)\right) = \lambda D\left(P_{Y|X}(\cdot|x)\|b_Y(\cdot)\right) - \lambda - \mu^b + \nu^b(x)$$

where $R := \sqrt{\sum_t \rho(t)b(t)}$. Here, $\alpha = |s_{01}|\sigma_0/\sqrt{k_0}$. We postulate that $X = \gamma_0 S, \widehat{S} = \gamma_1 Y$ to solve the KKT conditions of $\mathbf{C}_{\mathbf{B}'}$. Assume $\gamma_0 > 0$, so $R = \sqrt{k_0}\gamma_0\sigma_0$. As in Theorem 3.1, put $\nu^b(x), \nu^a(\widehat{s}|s) \equiv 0$, $\mu^b = \lambda c_1 - \lambda$, $\mu^a(s) \equiv -\lambda c_2(s)$ and compare coefficients to get $k_0 = \lambda/(2(1-(\alpha/2R))(\gamma_0^2\sigma_0^2 + \sigma_w^2))$, $\gamma_0\gamma_1\bar{P} = 1$ and $\lambda = 2\gamma_1^2\sigma_w^2\bar{P}$. Simplifying gives $\gamma_0 = \gamma_0^*$, where γ_0^* satisfies (20) and $\gamma_1 = \gamma_1^* = \xi_1(\gamma_0^*)$. The equality in DPI follows as in Theorem 3.1. Therefore, the distribution $(S, \gamma_0^*S, Y, \gamma_1^*Y)$ is optimal for $\mathbf{C}_{\mathbf{B}'}$ and, by Markoviantiy, for $\mathbf{B}'$. Thus, we get that

$$\mathrm{OPT}(\mathbf{B}') = \frac{\sigma_w^2\sigma_0^2}{\gamma_0^{*2}\sigma_0^2 + \sigma_w^2} + k_0\gamma_0^{*2}\sigma_0^2 - |s_{01}|\gamma_0^*\sigma_0^2.$$

Now consider the PDF Q^{**} defined by $(S, \gamma_0^{**}S, Y, \gamma_1^{**}Y)$, where $\gamma_0^{**} \in \mathbb{R}$ and $\gamma_1^{**} = \xi_1(\gamma_0^{**})$. It is easy to check if $\gamma_0^{**} = -\mathrm{sgn}(s_{01})\gamma_0^*$, the objective of $\mathbf{B}$ evaluated at Q^{**} equals $\mathrm{OPT}(\mathbf{B}')$. Thus Q^{**} solves $\mathbf{B}$. ■

Note that (20) agrees with (6) of Bansal and Başar with $P^* = \gamma_0^*\sigma_0$ and the γ_0^{**} agrees with the optimal controller they obtain in [2]. Although the above result did not utilize the convex relaxation of $\mathbf{B}$, it is indeed true that a solution of $\mathbf{B}$ also solves its convex relaxation with $\mathcal{Q}_{\rm DPI}$. Below we emphasize this.

Theorem 3.4: Let the setting and assumptions of Theorem 3.3 hold. Any solution of $\mathbf{B}$ solves its relaxation $\mathbf{C}_{\mathbf{B}}$ with $\mathcal{Q}_{\rm DPI}$. i.e., the optimal value of $\mathbf{B}$ equals the optimal value of $\mathbf{C}_{\mathbf{B}}$. In fact, $\mathrm{OPT}(\mathbf{B}) = \mathrm{OPT}(\mathbf{B}') = \mathrm{OPT}(\mathbf{C}_{\mathbf{B}}) = \mathrm{OPT}(\mathbf{C}_{\mathbf{B}'})$.

Proof: We only need to prove the last sequence of equalities. Since $\mathcal{Q} \subset \mathcal{Q}_{\rm DPI}$, $\mathrm{OPT}(\mathbf{B}) \geq \mathrm{OPT}(\mathbf{C}_{\mathbf{B}})$ and by Cauchy–Schwartz inequality, $\mathrm{OPT}(\mathbf{C}_{\mathbf{B}}) \geq \mathrm{OPT}(\mathbf{C}_{\mathbf{B}'})$, but Theorem 3.3 shows $\mathrm{OPT}(\mathbf{B}) = \mathrm{OPT}(\mathbf{C}_{\mathbf{B}'})$. Combining this gives the result. ■

We now show that as a consequence of Theorem 3.4, any solution of $\mathbf{B}$ must satisfy equality in the DPI. For this we need the following lemma.

Lemma 3.5: Let the setting and assumptions of Theorem 3.3 hold and let Q^* be a solution of $\mathbf{B}$. Then Q^* also solves

$$\min_{Q\in\mathcal{Q}_{\rm DPI}} \langle \delta + (1-\alpha/(2R^*))\,\rho, Q\rangle \qquad (\mathbf{C}_{\mathbf{G}'})$$

where $R^* = \sqrt{\langle \rho, Q^*\rangle}$.

Proof: From Theorem 3.4, Q^* solves $\mathbf{C}_{\mathbf{B}'}$. The proof now follows easily by comparison of the KKT conditions of $\mathbf{C}_{\mathbf{B}'}$ and $\mathbf{C}_{\mathbf{G}'}$. ■

Theorem 3.5: Let the setting and assumptions of Theorem 3.3 hold and let Q^* be a solution of $\mathbf{B}$. Define the rate-distortion function $R(\cdot)$ and capacity cost function $C(\cdot)$ as in (7), (8) for distortion δ and cost $(1-\alpha/(2R^*))\rho$, where $R^* = \sqrt{\langle \rho, Q^*\rangle}$. Suppose for each $P \geq 0$ there exists $D \geq 0$ such that $C(P) = R(D)$. Then the equality holds in the DPI under the distribution Q^*.

Proof: By Lemma 3.5, Q^* solves $\mathbf{C}_{\mathbf{G}'}$. Let $a = Q^*_{\widehat{S}|S}$, $b = Q^*_X$ and for sake of contradiction, suppose $I(aP_S) < I(P_{Y|X}b)$. Clearly, for $D = \mathbb{E}_a[\delta]$ and $P = \mathbb{E}_b[(1-\alpha/(2R^*))\rho]$, $R(D) \leq I(aP_S) < I(P_{Y|X}b) \leq C(P)$. By the hypothesis, there exists D' with $R(D') = C(P) > 0$ and since $R(\cdot)$ is a strictly decreasing when it is positive, $D' < D$. Let a' be a distribution on $\widehat{S}|S$ such that $D' = \mathbb{E}_{a'}[\delta]$. Then $Q'(z) \equiv P_S(s)a'(\widehat{s}|s)P_{Y|X}(y|x)b(x)$ is feasible for $\mathbf{C}_{\mathbf{G}'}$ and has a lower value than Q^*. A contradiction. ■

Remark III.6. DPI in Problem: $\mathbf{B}$: It is an open problem to ascertain if there a way of solving $\mathbf{B}$ that does not use the DPI [4]. The above theorem says that, except in a degenerate case, equality in the DPI is necessary for optimality of $\mathbf{B}$. Thus any other solution method must imply this equality. Note that this is true even for solutions of $\mathbf{B}$ that are not "linear" (where $X = \gamma_0 S$, $\widehat{S} = \gamma_1 Y$) or deterministic. □

The original proof of Bansal and Başar used Cauchy–Schwartz inequality (cf. Section II-A). We used this inequality in the construction of the auxiliary problem $\mathbf{B}'$. We now present an alternative proof by directly solving the convex relaxation $\mathbf{C_B}$, but unlike $\mathbf{C_{B'}}$ or $\mathbf{C_G}$, the objective of $\mathbf{C_B}$ cannot written in terms of a, b alone and we have to deal with the harder system $(\mathbf{KKT_{C_N}})$.

Proof of Theorem 3.3 (Without Cauchy-Schwartz): We will solve $(\mathbf{KKT_{C_N}})$ for $\mathbf{C_B}$. Let $X \sim \gamma_0 S$, $\widehat{S} \sim \gamma_1 Y$ and Q be the resulting PDF. In $(\mathbf{KKT_{C_N}})$, use (18), (19), and put $\gamma_1 = \xi_1(\gamma_0)$, $\lambda = 2\gamma_1^2\sigma_w^2\bar{P}$, $\mu^b = \lambda c_1 - \lambda$, $\nu^a(\cdot), \nu^b(\cdot), \lambda^p(\cdot) \equiv 0$, and $\mu^a(s) \equiv -\lambda c_2(s) + \bar{\mu}^a(s)$, where $\bar{\mu}^a(s)$ will be determined later. This reduces $(*)$ to

$$\nu(z) \equiv k_0 x^2 - \xi x^2 + s_{01} x s + \bar{\mu}^a(s)$$

where $\xi := \xi_0(\gamma_0)$. If $s_{01} = 0$, setting $k_0 = \xi$ and $\bar{\mu}^a(\cdot) \equiv 0$ solves the problem. So we assume $s_{01} \neq 0$. We limit our choice of γ_0 to those which satisfy $k_0 > \xi_0(\gamma_0)$ and first find $\bar{\mu}^a(s)$ so as to satisfy $\nu(\cdot) \geq 0$. Completing the squares gives

$$\nu(z) \equiv \left(\sqrt{k_0 - \xi}\, x + \frac{s_{01}}{2\sqrt{k_0 - \xi}} s\right)^2 - \frac{s_{01}^2 s^2}{4(k_0 - \xi)} + \bar{\mu}^a(s).$$

Put $\bar{\mu}^a(s) \equiv s_{01}^2 s^2/(4(k_0 - \xi))$, so that now $\nu(\cdot) \geq 0$. The complementarity slackness of the DPI constraint follows as before. So we only need to satisfy the complementarity of $\nu(\cdot)$ and $Q(\cdot)$. Observe that $\nu(z) > 0$ if and only if $x \neq -s_{01}s/(2(k_0 - \xi))$, whereas for $x \neq \gamma_0 s$, the PDF $Q(z) = 0$. Thus, it suffices to take $\gamma_0 = -s_{01}/(2(k_0 - \xi))$ to satisfy the complementarity of $\nu(\cdot)$ and $Q(\cdot)$. Therefore, taking $\gamma_0 = \gamma_0^{**} = -\mathrm{sgn}(s_{01})\gamma_0^*$, where γ_0^* is a positive solution of (20), satisfies $\gamma_0 = -s_{01}/(2(k_0 - \xi))$. The positivity of γ_0^* ensures that $k_0 > \xi$ since $s_{01} \neq 0$. This completes the proof. ■

Remark III.7. Separability of the Cost: Although the objective of $\mathbf{B}$ cannot be written in a, b alone, using Cauchy–Schwartz inequality, a problem $\mathbf{B}'$ with $\mathrm{OPT}(\mathbf{B}') = \mathrm{OPT}(\mathbf{B})$, and whose objective could be written in a, b was constructed. For a problem with this structure, results from communication theory apply more directly (due to the structure of the DPI) and Bansal and Başar were able to leverage them. The optimization approach, on the other hand, can address $\mathbf{B}$ more directly and through a more general framework, without the need for Cauchy–Schwartz inequality. □

In summary we have shown that $\mathbf{B}$ can be solved by a particular application of the broader concept of convex relaxation where the DPI was used for constructing the convex relaxation.

IV. Inverse Optimal Control

We now consider inverse optimal control and show how it may be addressed from the standpoint of convex relaxation. The specific problem we are interested in is problem $\mathbf{G}$, where $\kappa(z) \equiv \delta(s, \widehat{s}) + \rho(x)$, although the overall framework applies to any case of $\mathbf{N}$.

The thrust of this section is in showing that convex relaxations allow one to obtain a wider range of inverse optimal cost functions for $\mathbf{G}$ than are obtainable through the communication theoretic argument. To explain this, consider the convex relaxation $\mathbf{C_G}$ of $\mathbf{G}$ and recall the system $(\mathbf{KKT_{C_G}})$. We first clarify the distinction between inverse optimality for $\mathbf{G}$ and that for Gastpar's problem which is characterized by (9).

Remark IV.8. Inverse Optimal Control: Suppose we are interested in inverse optimality of δ and ρ for problem $\mathbf{G}$. Nonconvexity of $\mathcal{Q}$ makes this hard, but the normal cone of $\mathcal{Q}_{\mathrm{DPI}}$ provides a subset of the inverse optimal functions (cf. Section I-B). The right-hand side of the system $(\mathbf{KKT_{C_G}})$ is precisely this subset. Although the form of δ and ρ in $(\mathbf{KKT_{C_G}})$ are the same as Gastpar's in (9) (except for minor differences), ours are obtained for an altogether different notion of optimality. Gastpar has inverse optimality *over deterministic codes of arbitrary block length*; ours are for inverse optimality over $\mathcal{Q}$, i.e., over single-letter random codes. For inverse optimality for $\mathbf{G}$, $(\mathbf{KKT_{C_G}})$ is *not* a necessary condition (unlike for optimality over arbitrary block lengths where (9) is also necessary). They have coincided because *mutual information serves the dual role* of characterizing Gastpar's optimality and of convexifying $\mathcal{Q}$. The results of the previous sections demonstrate that for solving $\mathbf{B}$ or $\mathbf{G}$, only the latter role is important.

There are minor differences between $(\mathbf{KKT_{C_G}})$ and (9). In (9), the constants c_1, c_2 are allowed to be distinct, but in $(\mathbf{KKT_{C_G}})$ they are equal $(= \lambda)$. This is because Gastpar considers *Pareto* optimality; a Pareto optimal code minimizes $\delta + \alpha\rho$ for some $\alpha > 0$. We have fixed $\alpha = 1$. Gastpar requires $c_1, c_2 > 0$, whereas our λ may be zero. Gastpar has an additional condition that $R(D^*)$ and $C(P^*)$ cannot be held fixed while reducing the optimal D^* and P^*. This is equivalent, in our problem, to the sensitivity of the DPI constraint, which implies the strict positivity of λ. Equality in the DPI (analogous to $R(D^*) = C(P^*)$) is a necessary condition for optimality in $\mathbf{C_G}$ too (this follows from the proof of Theorem 3.5). Finally, Gastpar does not have a term corresponding to $\nu^a(\widehat{s}|s)$; he has deliberately dropped it [38] since when $\nu^a(\widehat{s}|s) > 0$, $\log(a(\widehat{s}|s)/a(\widehat{s}))$ is undefined. □

We have chosen to focus on problem $\mathbf{G}$ because for this problem the inverse optimal cost functions can be read off directly from the KKT conditions. In general for problem $\mathbf{N}$, one would have to obtain the cost functions by solving the system $(\mathbf{KKT_{C_N}})$. The difficulty of doing this was articulated in Remark III.3.

The main implication of Remark IV.8 is as follows. The convexifcation using $\mathcal{Q}_{\mathrm{DPI}}$ is only one convexification of $\mathcal{Q}$ that provided a specific subset of the inverse optimal cost functions of $\mathbf{G}$, namely the one that coincided with Gastpar's inverse optimal cost functions. One could potentially employ other convexifications and these would yield other subsets of inverse optimal cost functions of $\mathbf{G}$. In the following section we will employ a convexifcation using a generalization of mutual information via f-divergence to obtain a larger class of inverse optimal cost functions for $\mathbf{G}$.

A. Inverse Optimal Control Using f-Divergence

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that $f(1) = 0$. The f-divergence of a pair of distributions P, Q supported on the same finite set is given by $D_f(P\|Q) = \sum_x Q(x) f(P(x)/Q(x))$. The corresponding f-mutual information between random variables X, Y, first studied by Ziv and Zakai [39], is defined as

$$I_f(X;Y) := D_f\left(p_X(\cdot)p_Y(\cdot)\|p(\cdot,\cdot)\right)$$

where $p(\cdot,\cdot)$ is the joint distribution of the random variables X, Y and p_X, p_Y are marginals of X and Y. The K-L divergence is given by $D(P\|Q) = D_{-\log}(Q\|P)$, and therefore, mutual information is $I(X;Y) = I_{-\log}(X;Y)$.

f-divergence enjoys many of the properties of K-L divergence (see, e.g., [40]). In particular, the f-mutual information satisfies the data processing inequality [39] (for clarity, we call this the f-DPI).

Lemma 4.1: Suppose X, Y, Z are discrete random variables such that $X \to Y \to Z$. Then, $I_f(X;Y) \geq I_f(X;Z)$.

Similarly if $X \to Y \to Z$, $I_f(Y;Z) \geq I_f(X;Z)$ and if $W \to X \to Y \to Z$, then $I_f(X;Y) \geq I_f(W;Z)$. Just like the K-L divergence, $D_f(P\|Q)$ is convex in (P,Q) [40]. From this, we get the following lemma.

Lemma 4.2: Let $p(x,y)$ denote the joint distribution of random variables X, Y and $p(x)$ denote the marginal of X. For fixed $p(x)$, $I_f(X;Y)$ is a convex function of $p(y|x)$.

The convexity of the set $\mathcal{Q}_{\rm DPI}$ is due to the fact that the mutual information $I(X;Y)$ is convex in $p(y|x)$ for fixed $p(x)$ *and* it is concave in $p(x)$ for fixed $p(y|x)$. The former property is shared by f-mutual information. For mutual information, the latter property is enabled by the additive decomposition of mutual information $I(X;Y) = H(Y) - H(Y|X)$. This additivity does not, in general, hold for f-mutual information and therefore it is not immediately clear if $I_f(X;Y)$ is concave in $p(x)$ for given $p(y|x)$. In section, we will restrict ourselves to those convex functions f for which f-mutual information has this property.

Definition 4.1: A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the *saddle property* if f is convex, $f(1) = 0$ and if the f-mutual information $I_f(X;Y)$ is a concave function of $p(x)$ for any fixed $p(y|x)$.

Remark IV.9: $f(x) \equiv -\log(x)$ has the saddle property, so the set of such functions is not empty. Furthermore, this is not a vacuous definition: $f(x) \equiv c - dx$ for constants c, d also has the saddle property. □

Define the set

$$\mathcal{Q}_{f\text{-DPI}} = \Big\{Q \in \mathcal{P}(\mathcal{Z}) | Q_S = P_S, Q_{Y|X} = P_{Y|X}, \\ I_f(Q_{S,\widehat{S}}) \leq I_f(Q_{X,Y})\Big\}. \quad (21)$$

By Lemma 4.1, $\mathcal{Q} \subset \mathcal{Q}_{f\text{-DPI}}$. Furthermore, we have the following lemma.

Lemma 4.3: Let f have the saddle property. Then $\mathcal{Q}_{f\text{-DPI}}$ is convex.

Proof: Follows as in Lemma 3.3. ■

Consider the following relaxation of $\mathbf{N}$, denoted f-$\mathbf{C_N}$

$$\min_{Q\in\mathcal{Q}_{f\text{-DPI}}} \langle \kappa, Q\rangle. \qquad (f\text{-}\mathbf{C_N})$$

It follows that if f has the saddle property the relaxation f-$\mathbf{C_N}$ is a convex optimization problem. f-$\mathbf{C_N}$ can be written explicitly as a mathematical program in the variable $Q \in \mathcal{Q}_{f\text{-DPI}}$ and additional variables $a = Q_{\widehat{S}|S}$, $b = Q_X$, in the same way as $\mathbf{C_N}$ was written.

Now consider problem $\mathbf{G}$, i.e., take $\kappa(z) = \delta(s,\widehat{s}) + \rho(x)$. Then following the same line of arguments made for $\mathbf{C_G}$, the objective of the relaxation f-$\mathbf{C_G}$ can be written in terms of a, b alone and Q can be eliminated. The problem f-$\mathbf{C_G}$ is the same as $\mathbf{C_G}$, except that the constraint "$I(aP_S) \leq I(P_{Y|X}b)$" is replaced by "$I_f(aP_S) \leq I_f(P_{Y|X}b)$." Let $\mu^a, \mu^b, \lambda, \nu^a, \nu^b$ be the Lagrange multipliers of the constraints of problem f-$\mathbf{C_G}$. a, b are optimal for f-$\mathbf{C_G}$ if and only if together with the Lagrange multipliers, they satisfy, $\forall z \in \mathcal{Z}$

$$\delta(s,\widehat{s})P_S(s) = -\lambda\frac{\partial I_f(aP_S)}{\partial a(\widehat{s}|s)} + P_S(s)\mu^a(s) + P_S(s)\nu^a(\widehat{s}|s) \qquad (\mathbf{KKT}_{f\text{-}\mathbf{C_G}})$$

$$\rho(x) = \lambda\frac{\partial I_f(P_{Y|X}b)}{\partial b(x)} + \mu^b + \nu^b(x)$$

$$\lambda \geq 0, I(P_{Y|X}b) - I(aP_S) \geq 0,\ \lambda\left(I_f(P_{Y|X}b) - I_f(aP_S)\right) = 0$$

$$\sum_{\widehat{s}} a(\widehat{s}|s) = 1, \langle a(\cdot|s), \nu^a(\cdot|s)\rangle = 0, \nu^a(\widehat{s}|s) \geq 0,$$

$$\sum_x b(x) = 1,\ \langle b, \nu^b\rangle = 0, \nu^b(\cdot) \geq 0, b(\cdot) \geq 0, a(\widehat{s}|s) \geq 0.$$

Theorem 4.1: Let $Q \in \mathcal{Q}$ be a candidate solution of $\mathbf{G}$. Denote $a(\widehat{s}|s) = Q_{\widehat{S}|S}(\widehat{s}|s)$, $b(x) = Q_X(x)$. Then for this Q, the following functions δ, ρ are inverse optimal for $\mathbf{G}$:

$$\delta(s,\widehat{s}) = -\frac{1}{P_S(s)}\lambda\frac{\partial I_f(aP_S)}{\partial a(\widehat{s}|s)} + \mu^a(s) + \nu^a(\widehat{s}|s)$$

$$\rho(x) = \lambda\frac{\partial I_f(P_{Y|X}b)}{\partial b(x)} + \mu^b + \nu^b(x).$$

where f is any continuously differentiable function having the saddle property, $\lambda \geq 0$ is a scalar satisfying $\lambda(I_f(P_{Y|X}b) - I_f(aP_S)) = 0$ and the other parameters are constants or functions satisfying, $\mu^b \in \mathbb{R}, \nu^b : \mathcal{X} \to [0,\infty)$, $\mu^a : \mathcal{S} \to \mathbb{R}$, $\nu^a : \mathcal{S} \times \widehat{\mathcal{S}} \to [0,\infty)$, $\langle \nu^b, b\rangle = 0$, $\langle \nu^a(\cdot|s), a(\cdot|s)\rangle = 0$ for all s.

Proof: Under the given conditions, the system $(\mathbf{KKT}_{f\text{-}\mathbf{C_G}})$ is satisfied by Q. Furthermore, since f has the saddle property, f-$\mathbf{C_G}$ is a convex optimization problem. This implies that δ and ρ are inverse optimal for Q for problem f-$\mathbf{C_G}$. As a consequence, they are inverse optimal for $\mathbf{G}$. ■

Remark IV.10: Notice that the saddle property is not required to claim $\mathcal{Q} \subset \mathcal{Q}_{f\text{-DPI}}$. It is only required to claim that the Q that satisfies $(\mathbf{KKT}_{f\text{-}\mathbf{C_G}})$ is indeed optimal for f-$\mathbf{C_G}$, for which we need the convexity of f-$\mathbf{C_G}$. Finally, note that [39] develops the approach of getting tighter bounds on distortion for finite blocklength codes. Bansal and Başar [2] could have, in principle, applied this to their problem by replacing mutual information with f-mutual information. It is conceivable that this line of attack would be hindered by the lack of concavity in $I_f(P_{Y|X}b)$. □

Remark IV.11: The tightness of the relaxation $\mathcal{Q}_{f\text{-DPI}}$ depends on the choice of f. The tightest relaxation of $\mathcal{Q}$ obtainable is the *convex hull* of $\mathcal{Q}$. Characterizing this set remains a challenge. □

V. Convex Relaxation of the Witsenhausen Problem

We now study the relaxation with $\mathcal{Q}_{\rm DPI}$ of the Witsenhausen problem, i.e., $\mathbf{N}$ with $\kappa(z) \equiv \zeta(x,\widehat{s}) + \tau(x,s)$. Consider the problems

$$\min_{Q\in\mathcal{Q}} \langle \zeta+\tau, Q\rangle, \quad (\mathbf{W}) \quad \min_{Q\in\mathcal{Q}_{\rm DPI}} \langle \zeta+\tau, Q\rangle. \qquad (\mathbf{C_W})$$

The KKT conditions of $\mathbf{C_W}$ are a special case of $(\mathbf{KKT_{C_N}})$. In Remark III.3, we alluded to the fact that solving $(*)$ is harder for this problem. Although we cannot solve $\mathbf{C_W}$, we can characterize its optimal value in terms of the Lagrange multipliers $\mu^a(\cdot), \lambda, \mu^b$.

Theorem 5.1: Suppose $\mathbf{C_W}$ has a solution $Q = Q^*$ and suppose system $(\mathbf{KKT_{C_N}})$ characterizes the solution of $\mathbf{C_W}$. Then the optimal value of $\mathbf{C_W}$ is given by $-(\mathbb{E}[\mu^{a*}(S)] + \lambda^* + \mu^{b*})$, where $\mu^{a*}(\cdot), \lambda^*, \mu^{b*}$ are the values of the (scaled) Lagrange multipliers $\mu^a(\cdot), \lambda, \mu^b$ that satisfy $(\mathbf{KKT_{C_N}})$ for Q^*.

Proof: In equation $(*)$ of $(\mathbf{KKT_{C_N}})$ take expectation with Q^* to get $\mathbb{E}[\zeta(X,\widehat{S}) + \tau(X,S) + \mu^{a*}(S) + \lambda^* + \mu^{b*}] + \lambda(I(aP_S) - I(P_{Y|X}b)) = 0$, where the expectation is over Q^*. Complementarity slackness for the DPI gives $\mathbb{E}[\zeta(X,\widehat{S}) + \tau(X,S)] = -(\mathbb{E}[\mu^{a*}(S)] + \lambda^* + \mu^{b*})$. ■

$-(\mathbb{E}[\mu^{a*}(S)] + \lambda^* + \mu^{b*})$ is a lower bound on $\mathrm{OPT}(\mathbf{W})$ too. Notice that although μ^{a*} itself depends on Q^*, the expectation therein is over the *given* distribution P_S. Also note that the result applies for any cost structure and not only for $\mathbf{W}$.

We now show that it is not possible to solve $(\mathbf{KKT_{C_N}})$ for $\mathbf{C_W}$ with $\nu(\cdot) \equiv 0$ using linear controllers.

Theorem 5.2: Consider the convex relaxation of the Witsenhausen problem, i.e., in $\mathbf{C_W}$ suppose $S \sim \mathcal{N}(0,\sigma_0^2)$, $Y = X + W, W \sim \mathcal{N}(0,\sigma_w^2)$ and independent of $S, \zeta(x,\widehat{s}) \equiv (x-\widehat{s})^2, \tau(x,s) \equiv (x-s)^2$. There exist no constants γ_0, γ_1 such that the PDF $(S, \gamma_0 S, Y, \gamma_1 Y)$ solves $(\mathbf{KKT_{C_N}})$ with $\nu(\cdot) \equiv 0$.

Proof: Assume the contrary, i.e., assume $(\mathbf{KKT_{C_N}})$ holds for this PDF. The distribution $a(\widehat{s}|s)$ induced by γ_0, γ_1 is positive for all $s, \widehat{s}$. Since $\langle a(\cdot|s), \nu^a(\cdot|s)\rangle = 0$ for all s, $\nu^a(\widehat{s}|s) = 0$ for almost every $\widehat{s}, s$. Likewise $\nu^b(x) = 0$ for almost every x. Now in $(*)$, put $\nu(\cdot) = 0$ and take expectation with $P_{Y|X}(\cdot|x)$ to get that for almost every z, $(x-\widehat{s})^2 + (x-s)^2 + f_1(x) + f_2(s,\widehat{s}) + \mu^a(s) + c = 0$, where $f_1(\cdot), f_2(\cdot)$ are quadratic polynomials, $\mu^a(\cdot)$ is a function on $\mathcal{S}$ and c is a constant. This implies that $(x-\widehat{s})^2 + f_2(s,\widehat{s})$ is independent of $\widehat{s}$ for all x, s except from a set of measure zero. Clearly, this is not possible for a quadratic f_2. ■

Remark V.12. Linear Controllers for the Witsenhausen Problem: In a sense, the above theorem was only a sanity check. We already know from [3] that linear controllers are not optimal for the Witsenhausen problem; had we found constants γ_0, γ_1 that solved $\mathbf{C_W}$, they would automatically be optimal for the Witsenhausen problem. □

This indicates that if at all a solution of the Witsenhausen problem is to be found through $\mathbf{C_W}$, finding it will require a sophisticated attempt at solving $(\mathbf{KKT_{C_N}})$. Or, it may require a different convexification.

VI. Conclusion

This paper has presented an optimization-based approach and a general framework for stochastic control problems with nonclassical information structure that employed the idea of convex relaxation. We recovered solutions obtained by Bansal and Başar through this approach. In our formulation, these stochastic control problems are nonconvex optimization problems and the information-theoretic data processing inequality appears as an artifice of convexifying them. Using certain f-divergences we obtained a wider set of inverse optimal cost functions than were obtainable through information theoretic methods. We gave insights into the relation of the cost structure of the problem and the structure of the convexification for inverse optimal control.

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